Special Relativity and Magnetism in an Introductory Physics Course

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Too often, students in introductory courses are left with the impression that Einstein's special theory of relativity comes into play only when the relative speed of two objects is an appreciable fraction of the speed of light (*c*). In fact, relativistic length contraction, along with Coulomb's law, accounts quantitatively for the force on a charged particle as it moves relative to a current-carrying wire. That force, which in the reference frame of the wire we call "magnetic," is measurable and important even at relative speeds on the order of $10^{-12}c$. This paper offers a straightforward way of introducing students to the connection between magnetism and special relativity and provides references to more in-depth treatments, especially those of E.R. Huggins[1] and of E.M. Purcell as simplified by D.V. Schroeder.[2]

The Short Version

Suppose a conventional (positive) current flows in a wire that is stationary in the laboratory reference frame. To us the wire is electrically neutral, the numbers of positive and negative charges per unit length of wire being equal. But what does the wire "look like" to a positively charged particle moving in the same direction and at the same speed as the charge moving in the wire? To such a particle, the positive charges in the wire are stationary while the negative charges are moving. To the particle, is the wire still neutral?

Einstein's special theory of relativity tells us that relative motion causes a shortening (contraction) of space along the direction of motion.[3] Thus, in the reference frame of the moving positively charged particle, the average distance separating negative charges in the wire is smaller than the average distance separating positive charges. Consequently, in the reference frame of the moving particle, the wire is negatively charged. The resulting Coulombic attraction is what we in the lab frame normally refer to as the magnetic force.

This simple, qualitative explanation should provoke a number of questions. Why, for example, doesn't the relative motion of positive and negative charges cause the wire to appear charged in the laboratory frame? Moreover, the drift speed with which charge moves along a wire is a tiny fraction of the speed of light. Can we really account for magnetic forces on the basis of what must be a truly minute relativistic effect?

What follows is intended to answer these questions and to encourage teachers to include Einstein's explanation of magnetism in introductory physics courses.[4] More in-depth treatments are of course available, to which the present author is greatly indebted.[5]

One Step at a Time

Suppose a wire, stationary in the laboratory reference frame, carries a steady electric current. The net linear charge density along the wire (λ) is the algebraic sum of the linear densities of positive and negative charges, which we write as

$$\lambda = (e/L^{+} - e/L^{-}_{0}). \qquad (1)$$

In this equation, e is the elementary charge (1.60 ×

10^{-19} C), L^+ is the average distance between positive elementary charges, and L^-_0 is the average distance between negative elementary charges.

Why the subscript to L^-_0? In relativistic terms, L^-_0 is a "proper length," i.e., it is the distance between objects that are at rest relative to the reference frame in which the distance is measured. We are following the convention that current is due to the motion of positive charges, so the charges at rest in the lab frame are negative. The length L^+ is the distance between the positive charges, objects that are moving relative to the laboratory frame in which the distance is measured. It is not a proper length. Accordingly, L^+ is subject to relativistic length contraction according to

$$L^+ = L^+_0 / \gamma, \tag{2}$$

where L^+_0 is the distance between positive charges measured in their own reference frame. As usual in relativity theory,

$$\gamma = 1/\sqrt{1-(v^2/c^2)},$$

where v is relative speed of the two reference frames. Here, that is the "drift" speed of the charge carriers in the wire.[6] While the current flows, the quantity γ is slightly greater than one, so the distance between the moving positive charges is "Lorentz-contracted" from L^+_0 to L^+.[7]

Experimentally we know the wire is electrically neutral in the laboratory frame. Thus $\lambda = 0$ and, as a consequence, L^+ must equal L^-_0. That is, the Lorentz-contracted length separating moving positive charges is equal to the proper length separating the stationary negative charges. So we can write with confidence

$$L^+ = L^-_0. \tag{3}$$

Now consider how the wire appears to a positively charged particle moving parallel to the wire at the same speed as the positive charge in the wire. In the reference frame of such a particle, the positive charge in the wire is stationary and therefore not subject to relativistic length contraction. So the distance separating positive charge in this frame is not L^+ but the slightly larger proper length L^+_0.

The negative charges in the wire, stationary in the lab frame, also look different to the moving positive particle. They appear to be moving, and in the reference frame of the positive particle, the distance separating negative charge in the wire is not L^-_0 but the slightly smaller L^-, where

$$L^- = L^-_0/ \gamma. \tag{4}$$

The net charge density of the wire in the moving reference frame of the charged particle, which we will call λ', is the algebraic sum of positive and negative charge densities in that moving frame, or

$$\begin{aligned}\lambda' &= (e / L^+_0 - e / L^-)\\ &= e(1/ L^+_0 - 1/L^-).\end{aligned}$$

But $L^+_0 = \gamma L^+$, $L^+ = L^-_0$ (our neutrality condition in the laboratory frame), and $L^-_0 = \gamma L^-$, so

$$L^+_0 = \gamma^2 L^-. \tag{5}$$

Substituting for L^+_0 and replacing $1/\gamma^2$ with $1 - v^2/c^2$, we obtain

$$\begin{aligned}\lambda' &= e(1/\gamma^2 L^- - 1/L^-)\\ &= (e / L^-)(1/ \gamma^2 - 1)\\ &= (e / L^-)(-v^2/c^2),\end{aligned} \tag{6}$$

a quantity that has a small negative value. Thus, to the moving positively charged particle, the wire appears to have a net negative linear charge density. This net charge density is not neutralized by the other circuit elements because those entities are stationary in the lab frame. They balance L^+ and L^-_0 , not L^- and L^+_0. But it is precisely L^- and L^+_0 that matter to the moving particle. Because those densities are unequal, the particle is attracted to the wire by a Coulombic force.

Let's Get Quantitative

Can we really account quantitatively for magnetic forces on the basis of special relativity? From Gauss's law, we know the magnitude of the force on a charged particle near a wire with linear charge density λ':

$$F_E' = q\lambda'/(2\pi \varepsilon_0 r), \tag{7}$$

where q is the charge on the particle, r is the distance between the particle and the wire, and ε_0 is the electric permittivity of empty space (approx. 8.85×10^{-12} $C^2/N{\cdot}m^2$). In Eq. (7) both force and charge density carry a prime to indicate they are quantities measured in the reference frame of the moving charged particle.

From Ampere's law we know the magnitude of the attractive force on a charge q, a distance r from a wire carrying a current i, moving with velocity (v) in the same direction as the charge in the wire:

$$F_B = qv\,\mu_0 i\,/(2\pi r). \tag{8}$$

Note none of the quantities in Eq. (8) are primed since they are all measured in the lab frame. Based upon Eq. (6) what can we conclude about the relationship between F_E' and F_B?

In the frame of the moving positively charged particle, the current in the wire is due to the motion of negative charge. The magnitude of that current (i') is the drift speed (v) multiplied by the linear density of negative charge in the wire, e/L^-. That is,

$$i' = v(e/L^-),$$
or
$$e/L^- = i'/v. \tag{9}$$

Substituting i'/v for e/L^- in Eq. (6) and $1/\sqrt{\mu_0\varepsilon_0}$ for c, we find

$$\begin{aligned}\lambda' &= -(i'/v)(v^2/c^2)\\ &= -i'v/c^2\\ &= -\mu_0\,\varepsilon_0\,i'v.\end{aligned} \tag{10}$$

For values of v small in comparison to c, i' and i are essentially equal, allowing us to write:[9]

$$\lambda' = -\mu_0\,\varepsilon_0\,iv.$$

Substituting this value for λ' in Eq. (7) we have for the magnitude of the Coulomb force

$$\begin{aligned}F_E' &= q\mu_0\varepsilon_0 iv/(2\pi\varepsilon_0 r)\\ &= qv\mu_0 i\,/(2\pi r).\end{aligned} \tag{11}$$

Comparing Eqs. (8) and (11) we obtain

$$F_B = F_E'.$$

Thus, relativistic length contraction along with Coulomb's law accounts quantitatively for the attractive force between a moving positively charged particle and a wire carrying a positive current in the same direction.[10]

From a Moving Particle to a Second Wire

It's a small step from here to explain the force between two parallel wires carrying current in the same direction. From our study of electromagnetism we know that force to be

$$F_B = \mu_0 i^2\,l\,/\,(2\pi r), \tag{12}$$

where i is the current in each wire and l is their shared length. But the current in each wire is just

$$i = v(q/l), \tag{13}$$

where q is the amount of charge moving with drift speed v in a length of wire l. Substituting $qv = il$ into Eq. (11), we obtain

$$F_E' = \mu_0 i^2 l\,/(2\pi r), \tag{14}$$

as intended. Again, we have accounted quantitatively for observed magnetic forces through relativistic length contraction and Coulomb's law, subject only to the limitation that v be much less than c.

How Could this Possibly be True?

Attentive students are astonished to learn that speeds on the order of 10^{-4} m/s can have relativistic effects. As Huggins so vividly points out, a miniscule relativistic contraction becomes important because electrostatic forces, per coulomb of unbalanced charge, are so enormous.[11] Just how enormous can be shown by estimating the electrostatic imbalance it would take to produce the observed attraction between two current-carrying wires. We start by replacing q in Eq. (7) with $\lambda' l$ to obtain the attractive force between two wires having the same linear charge density λ':

$$F_E' = \lambda'^2 l/(2\pi\varepsilon_0 r). \quad (15)$$

Equating the left-hand sides of Eqs. (12) and (15), we obtain

$$\mu_0 i^2 l/(2\pi r) = \lambda'^2 l/(2\pi\varepsilon_0 r)$$
$$\mu_0 i^2 = \lambda'^2/(\varepsilon_0)$$
$$i^2/\lambda'^2 = 1/(\mu_0\,\varepsilon_0) \quad (16)$$
$$i/\lambda' = 1/\sqrt{\mu_0\varepsilon_0}$$
$$= c$$

or

$$\lambda' = i/c.$$

This intriguing result tells us that the attractive force between two wires, each carrying 1 A of current, is about the same as that between two wires having equal and opposite net linear charge densities of 3.3×10^{-9} C/m. The linear density of conduction electrons in a 1-mm-diameter copper wire is about 1.1×10^4 C/m.[12] Thus, the net linear charge density needed to account for the force between the wires is about three parts in 10^{13} of the total mobile charge in the wire. This is the effective charge imbalance explained by relativistic length contraction.[13]

Discussion

Special relativity also explains the repulsive force on a positively charged particle when such a particle moves in a direction opposite to the direction of the conventional current. In brief, the charged particle sees a wire in which both positive and negative charges move in the same direction. The distances between both positive and negative charges are Lorentz-contracted, but not by the same amount since the positive charges will be moving at twice the speed of the negative charges and γ is not linearly related to v. For a thorough account, as well as an explanation of the force on a charge moving perpendicular to the wire, see D.V. Schroeder's excellent online materials.[14]

Hopefully this paper has offered the reader a choice of pedagogical approaches, varying in mathematical detail, to make students in introductory physics courses aware of this most tangible application of the special theory of relativity. In doing so we might keep in mind Einstein's observation that

> "the special theory of relativity owes its origin principally to Maxwell's theory of the electromagnetic field. … On the other hand, the services tendered by the special theory of relativity to its parent, Maxwell's theory of the electromagnetic field, are less adequately recognized. Up to that time the electric field and the magnetic field were regarded as existing separately even if a close causal correlation between the two types of field was provided by Maxwell's field equations. But the special theory of relativity showed that this causal correlation corresponds to an essential identity of the two types of field."[15]

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References

1. E.R. Huggins, *Physics 1* (W.A. Benjamin, 1968); and E.R. Huggins, *Physics 2000*, available in hard copy or on CD through http://www.physics2000.com.
2. E.M. Purcell, *Electricity and Magnetism: Berkeley Physics Course* Vol. 2, 2nd ed. (McGraw-Hill, Boston, 1985); D.V. Schroeder, *Purcell Simplified, or Magnetism, Radiation, and Relativity* (summary of a talk presented at the 1999 Winter Meeting of the American Association of Physics Teachers, Anaheim, CA, Jan. 14, 1999); and D.V. Schroeder, *Magnetism, Radiation, and Relativity*, 1999. The latter two references are available in full through http://departments.weber.edu/physics/schroeder/mrr/MRR.html.
3. The elements of special relativity theory needed here can be found in most introductory physics texts. Other especially helpful resources are: A.B. Arons, *Teaching Introductory Physics* (Wiley, New York, 1997) Part I, pp. 301-308; and W. Schneider, *Maxwell's Conundrum: A Serious But Not Ponderous Book About Relativity* (Cavendish Press, Ann Arbor, MI, 2000).
4. The author has taught this material to high school students in both algebra-based and calculus-based courses. Comprehension has been sufficiently high that most are at least able to answer straightforward multiple-choice questions on the topic. A few introductory texts actually introduce magnetism as a relativistic effect. See, e.g., Huggins, *Physics 2000* and R.D. Knight, *Physics for Scientists and Engineers* (Benjamin Cummings, 2004).

5. In addition to the cited publications of E.R. Huggins, E.M. Purcell, and D.V. Schroeder, see A. Beiser, *Concepts of Modern Physics*, 6th ed. (McGraw-Hill, Boston, 2002); and D.J. Craik, *Electricity, Relativity and Magnetism: A Unified Text* (Wiley, New York, 1999).
6. Drift speeds are typically on the order of 10^{-4} m/s. This is many orders of magnitude less than the speed associated with the random, thermal motion of electrons; however, the latter is of no consequence here as it makes no contribution to the net motion of charge along the length of the wire.
7. In typical illustrations, the length of an object such as a rod or spaceship is subject to relativistic contraction. But such objects are composed of particles, and if the object as a whole is to contract, so must the distance between its constituent particles. The same reasoning applies to the average distance between charged particles in a wire.
8. To understand the physical basis of this identity, consider the conductive elements in the laboratory frame, including the emf device that maintains current flow. If in the laboratory frame the distances L^-_0 and L^+ were not equal, say L^+ being smaller, the wire would carry a net positive charge, which would drive positive charge out of the wire into the emf device, establishing electrical neutrality. See also P.C. Peters, "In what frame is a current-carrying conductor neutral?" *Am. J. Phys.* **53**, 1165–1169 (Dec. 1985).
9. The magnitude of the current in the lab frame (i) is the drift speed multiplied by the linear density of positive charge in the lab frame, that is, $i = v(e/L^+)$. From the reasoning that led to Eq. (5), we can conclude $i = v(e/L_0^-) = (v/\gamma)(e/L^-) = i'/\gamma$. Expansion of $v/c\gamma$ yields $v/c - (1/2)(v^3/c^3)$, disregarding higher-order terms. Thus, v/γ can be approximated by v (and i' by i) for a wide range of values of v for which (v^3/c^3) is negligible compared with v.
10. If students have already been introduced to relativistic mass, the approximation in the previous footnote can be avoided altogether. First, replace i' in Eq. (10) with γi, leading to the identity $F_B = F_E'/\gamma$. Then note that $F_E' = m(dy^2/dt'^2)$, where y is the direction perpendicular to the current and m is the particle rest mass. Due to time dilation, $dt' = dt/\gamma$, so $F_B = \gamma m(dy^2/dt^2)$. Since forces in the lab frame will be related to acceleration of the moving particle by the relativistic particle mass γm, the right-hand side of this equation is precisely equal to F_E.
11. E.R. Huggins, *Physics 2000*, p. 28-2 ff.
12. Such a wire has a cross-sectional area of 0.79×10^{-6} m^2. Copper has a density of conduction electrons of approximately 8.5×10^{28} m^{-3} or about 1.4×10^{10} C/m^3. For details, see D. Halliday, R. Resnick ,and J. Walker, *Fundamentals of Physics*, 6th ed. (Wiley, New York, 2001) p. 617.
13. The drift speed of the charge carriers in this example is the current density in A/m^2 divided by the charge density in C/m^3: 1 A/(7.9×10^{-7} m^2 $\times$ 1.4×10^{10} C/m^3) or approximately 9.0×10^{-5} m/s. The associated value of γ exceeds unity by 4.5×10^{-26}.
14. D.V. Schroeder, *Magnetism, Radiation, and Relativity,* 1999, p. 13 ff.
15. A. Einstein, "The History of Field Theory (The Olds and News of Field Theory)" original manuscript in the Karpeles Manuscript Library Museum, available in translation at http://www.rain.org/~karpeles/einsteindis.html.

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Going Fast!

"In 2003, the biologist Jeffrey Dukes calculated that the fossil fuels we burn in one year were made from organic matter 'containing 44 E(18) grams of carbon, which is more than 400 times the net primary [annual] productivity of the planet's current biota.' In plain English, this means that every year we use four centuries' worth of plants and animals."[1]

1. *Sustainable Population Australia Newsletter*, No. 69, (March 2006), p. 7.