4.12 SU(2), SU(3), AND NUCLEAR PARTICLES

The application of group theory to "elementary" particles has been labeled by Wigner the third stage of group theory and physics. The first stage was the search for the 32 point groups and the 230 space groups giving crystal symmetries—Section 4.9. The second stage was a search for representations such as the representations of O_3^+ and SU(2)—Section 4.10. Now in this third stage, physicists are back to a search for groups.

In discussing the strongly interacting particles of high energy physics and the special unitary groups SU(2) and SU(3), we should look to angular momentum and the rotation group O_3^+ for an analogy. Suppose we have an electron in the spherically symmetric attractive potential of some atomic nucleus. The electron's Schrödinger wavefunction may be characterized by three quantum numbers n, l, and m. The energy, however, is $2l + 1$-fold degenerate, depending only on n and l^1. The reason for this degeneracy may be stated in two equivalent ways:

1. The potential is spherically symmetric, independent of θ and φ, and
2. The Schrödinger Hamiltonian $-(\hbar^2/2m_e)\nabla^2 + V(r)$ is *invariant* under ordinary spacial rotations (O_3^+).

As a consequence of the spherical symmetry of the potential, the angular momentum **L** is conserved. In Section 4.11 the cartesian components of **L** are identified as the generators of the rotation group O_3^+. Instead of representing L_x, L_y, and L_z by operators, let us use matrices. The exercises at the end of Section 4.2 provide examples for $l = \frac{1}{2}$, 1, and $\frac{3}{2}$. The L_i matrices are $(2l + 1) \times (2l + 1)$ matrices with the dimension the same as the number of the degenerate states.[2] These L_i matrices generate the $(2l + 1) \times (2l + 1)$ irreducible representations of O_3^+. The dimension $2l + 1$ is identified with the $2l + 1$ degenerate states.

The common method of eliminating this degeneracy is to introduce a constant magnetic induction **B**. This leads to the Zeeman effect. This magnetic induction adds a term to the Schrödinger Hamiltonian that is *not* invariant under O_3^+. This is a symmetry-breaking term.

So much for the analogy. In the case of the strongly interacting particles (neutrons, protons, etc.) we cannot follow the analogy directly, because we do not yet fully understand the nuclear interaction. We do not know the Hamiltonian. So instead, let us run the analogy backward.

In the 1930s Heisenberg proposed that nuclear forces were charge-independent, that the only two massive particles (baryons) known then, the neutron and proton, were two different states of the *same* particle. Table 4.2 shows that they have almost the same mass. The fractional difference, $(m_n - m_p)/m_p \approx 0.0014$, is small, suggesting that the mass difference is produced by a small charge-dependent perturbation. It was convenient to describe this near degeneracy by introducing a quantity **I** with z-projections $I_3 = \frac{1}{2}$ for the proton, $-\frac{1}{2}$ for the neutron. The name coined for **I** was isospin. Isospin had nothing to do with spin (the particle's intrinsic angular momentum) but the two-

TABLE 4.3
Baryons with Spin $\frac{1}{2}$ Even Parity

		Mass (MeV)	Y	I	I_3
Ξ	Ξ^-	1321.300	-1	$\frac{1}{2}$	$-\frac{1}{2}$
	Ξ^0	1314.900			$+\frac{1}{2}$
Σ	Σ^-	1197.410	0	1	-1
	Σ^0	1192.540			0
	Σ^+	1189.470			$+1$
Λ	Λ	1115.500	0	0	0
N	n	939.550	1	$\frac{1}{2}$	$-\frac{1}{2}$
	p	938.256			$+\frac{1}{2}$

component isospin state vector obeyed the same mathematical relations as the spin $J = \frac{1}{2}$ state vector, and in particular could be taken to be an eigenvector of the Pauli σ_3 matrix.

In the absence of charge-dependent forces, isospin is conserved (the proton and neutron have the same mass) and we have a twofold degeneracy. Equivalently, the unknown nuclear Hamiltonian must be invariant under the group generated by the isospin matrices. The isospin matrices are just the three Pauli matrices (2×2 matrices), and the group generated is the SU(2) group of Section 4.10, also 2×2 corresponding to our twofold degeneracy.

By 1961 many more particles had been discovered (or created). The eight shown in Table 4.3 attracted particular attention.[3] It was convenient to describe them by characteristic quantum numbers, I for isospin, and Y for hypercharge. The particles may be grouped into charge or isospin multiplets. Then the hypercharge Y may be taken as twice the average charge of the multiplet. For the neutron–proton multiplet

$$Y = 2 \cdot \tfrac{1}{2}(0 + 1) = 1. \tag{4.289}$$

The hypercharge and isospin values are listed in Table 4.3.

From scattering and production experiments it had become clear that both hypercharge Y and isospin I were conserved under stong (nuclear) interaction. Remember L (or l) is conserved under a spherically symmetric Hamiltonian. The eight particles thus appeared as an eightfold degeneracy, but now with *two* quantities to be conserved. In 1961 Gell-Mann, and independently Ne'eman, suggested that the strong interaction should be invariant under the three-dimensional special unitary group, SU(3), that is, should have SU(3) symmetry.

The choice of SU(3) was based first on the existence of two conserved quantities. This dictated a group of rank 2, a group, two of whose generators

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[3] All masses are given in energy units, MeV.

(and only two) commuted. Second, the group had to have an 8×8 representation to account for the eight degenerate baryons. In a sense SU(3) was the simplest generalization of SU(2). Gell-Mann set up eight generators: three for the components of isospin, one for hypercharge, and four additional ones. All are 3×3, zero-trace matrices. As with O_3^+ and SU(2), there are an infinity of irreducible representations. An eight-dimensional one was associated with the eight particles of Table 4.3.[4]

We imagine the Hamiltonian for our eight baryons to be composed of three parts

$$H = H_{\text{strong}} + H_{\text{medium}} + H_{\text{electromagnetic}}. \tag{4.290}$$

The first part, H_{strong}, possesses the SU(3) symmetry and leads to the eightfold degeneracy. Introduction of a symmetry breaking interaction, H_{medium}, removes part of the degeneracy giving the four isospin multiples Ξ, Σ, Λ, and N. These are multiplets because H_{medium} still possesses SU(2) symmetry. Finally, the presence of charge-dependent forces splits the isospin multiplets and removes the last degeneracy. This imagined sequence is shown in Fig. 4.16.

Applying first-order perturbation theory of quantum mechanics, simple relations among the baryon masses may be calculated. Also, intensity rules for decay and scattering processes may be obtained.

Perhaps the most spectacular success of this SU(3) model has been its prediction of new particles. In 1961 four K and three π mesons (all pseudoscalar; spin 0, odd parity) suggested another octet, similar to the baryon octet. The SU(3) theory predicted an eighth meson η^0, mass 563 MeV. The η^0 meson, experimentally determined mass 548 MeV, was found soon after. Groupings of nine of the heavier baryons (all with spin $\frac{3}{2}$, even parity) suggested a 10-member group or decuplet. The missing tenth baryon was predicted to have a mass of about 1680 MeV and a negative charge. In 1964 the negatively charged Ω^-, mass 1675 ± 12 MeV, was discovered.

Since the completion of this $\frac{3}{2}^+$ decuplet, a $\frac{5}{2}^-$ (odd parity) multiplet for baryons and 1^- and 2^+ multiplets for mesons have been established.

The application of group theory to strongly interacting particles has been extended beyond SU(3). There has been an extensive investigation of SU(6) and of the more complex, higher-dimensional groups. Great attention has been paid to the group generators and to the *structure constants* in the generator commutation relations (such as $i\varepsilon_{ijk}$ for orbital angular momentum). These structure constants define a *Lie algebra*. It is possible to associate space integrals of current densities with the group generators. This leads to a current algebra far beyond the scope of this discussion.

To keep group theory and its very real accomplishment in proper perspective, we should emphasize that group theory identifies and formalizes symmetries.

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[4] This application of SU(3) has been called by Gell-Mann the "eightfold way." Note the eight independent parameters of SU(3) (from $n^2 - 1$), the eight generators, the 8×8 representation associated with eight particles. The name also refers to the Eightfold Way of Buddha.

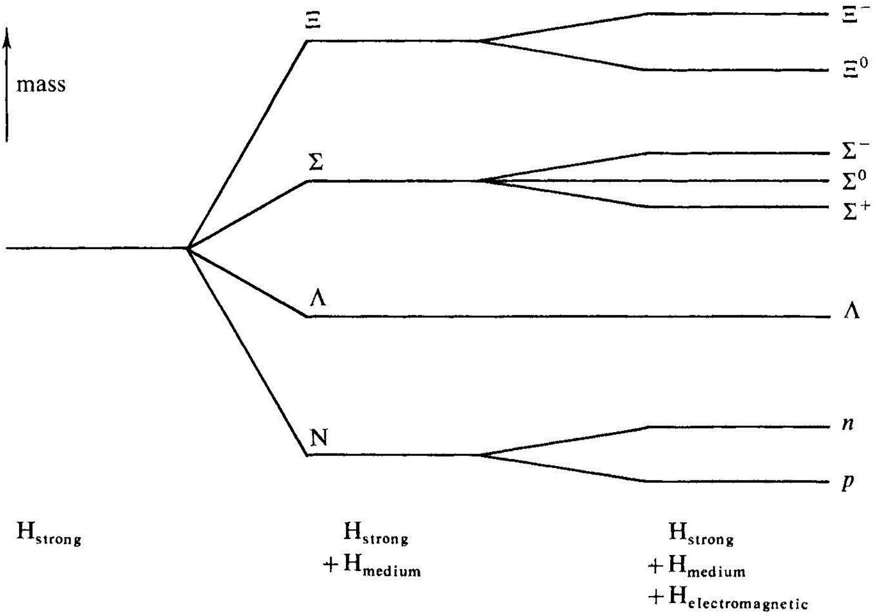
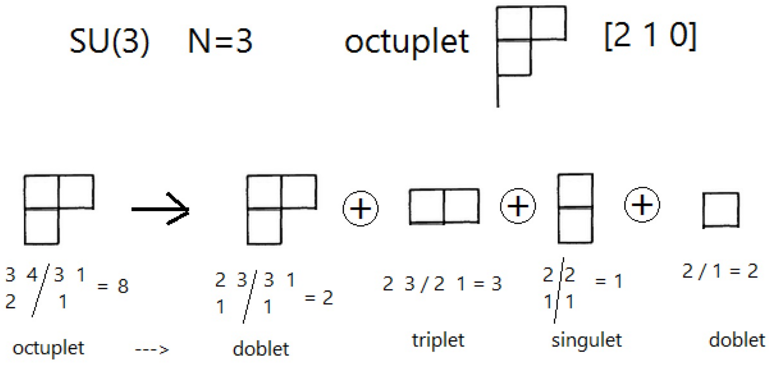


FIG. 4.16 Baryon mass splitting

It classifies (and sometimes predicts) particles. But aside from saying that one part of the Hamiltonian has SU(2) symmetry and another part has SU(3) symmetry, group theory says *nothing* about the particle interactions. Remember that the statement that the atomic potential is spherically symmetric tells us nothing about the radial dependence of the potential or of the wavefunction.



Josef Paldus   The Journal of Chemical Physics, Vol. 61, 5321 (1974)

II. IRREDUCIBLE REPRESENTATIONS OF THE UNITARY GROUP AND ITS RELEVANCE TO THE MANY BODY PROBLEM

Let us first present the most relevant facts of the representation theory of the unitary group $U(n)$ [or, in fact, the general linear group $GL(n)^{15}$].

Each finite-dimensional irreducible representation $\Gamma(m_n)$ of $U(n)$ may be uniquely specified by n ordered integers m_{in}, $(i=1,\ldots,n)$, called the highest weight of the representation, which we shall write as components of a vector m_n,

$$m_n = (m_{1n}, m_{2n}, \ldots, m_{nn}) \ , \tag{1}$$

where

$$m_{1n} \geqslant m_{2n} \geqslant \cdots \geqslant m_{nn} \ . \tag{2}$$

The dimension of this irreducible representation $\Gamma(m_n)$ is given by Weyl's dimension formula

$$\mathrm{Dim}[\Gamma(m_n)] = \prod_{i<j} (m_{in} - m_{jn} + j - i)/1!2!\cdots(n-1)! \ . \tag{3}$$

The individual vectors of the orthonormal canonical basis of the carrier space of this irreducible representation are uniquely specified by the triangular Gelfand patterns (or tableaus)

$$(m) \equiv \left[\begin{array}{c} m_{1n} \quad m_{2n} \quad m_{3n} \quad \cdots\cdots\cdots\cdots \quad m_{nn} \\ m_{1,n-1} \quad m_{2,n-1} \quad \cdots\cdots \quad m_{n-1,n-1} \\ m_{1,n-2} \quad \cdots\cdots \quad m_{n-2,n-2} \\ \cdots\cdot \\ m_{12} \quad m_{22} \\ m_{11} \end{array}\right] , \tag{4}$$

in which the first row contains the components of the highest weight vector (1) specifying a given irreducible representation $\Gamma(m_n)$, and the integers in the remaining $(n-1)$ rows, specifying uniquely a given basis vector, satisfy the so called "betweenness" conditions[18]

$$m_{i,j+1} \geq m_{i,j} \geq m_{i+1,j+1} \ , \tag{5}$$

$$i \leq j = 1, \ldots, (n-1) \quad .$$

Thus, the ith row of the Gelfand tableau consists of i nonincreasing integers,

$$m_{1i} \geq m_{2i} \geq \cdots \geq m_{ii} \ , \ (i=1,\ldots,n-1) \ , \tag{6}$$

the range of each being bounded by the integers appearing immediately above it in the tableau as expressed by the betweenness conditions (5). Each Gelfand pattern satisfying these conditions is called a "lexical" Gelfand pattern.

The ordering of these basis vectors (so-called lexical ordering) is defined as follows. Writing all the entries of the Gelfand pattern (m) as a row vector with $n(n+1)/2$ components, we define

$$p(m) \equiv (m_{1n}, \ldots, m_{nn}, m_{1,n-1}, \ldots, m_{n-1,n-1}, m_{1,n-2}, \ldots, m_{11}) \tag{7}$$

and consider the vector labeled by (m) to precede the vector (m') if the first nonvanishing component in the difference $p(m) - p(m')$ is a positive integer.

Thus, starting with the highest weight vector specifying a given irreducible representation $\Gamma(m_n)$, we can easily generate all lexical Gelfand tableaus by entering the integers, satisfying the betweenness conditions (5), into all the subsequent rows. Always using the largest possible integer first, we shall get the basis vectors automatically in the lexical order. The number of these vectors is, of course, given by Weyl's dimension formula (3).

The infinitesimal generators E_{ij} of $U(n)$ [or, in fact, of the general linear group $GL(n)$] are given by the following commutation relations (defining the structure constants of the corresponding Lie algebra)

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj} \quad , \tag{8}$$
$$(i, j, k, l = 1, \ldots, n) \quad .$$

In the case of the unitary group $U(n)$, these generators also satisfy the Hermitian conjugate relation

$$E_{ij}^{\dagger} = E_{ji} \quad . \tag{9}$$

The diagonal generators E_{ii} are called weight generators, while the off-diagonal ones are classified into the raising $(i<j)$ and lowering $(i>j)$ generators E_{ij}. This classification relies on the fact that the matrix representatives of these generators in the canonical (lexically ordered) Gelfand–Tsetlin basis are diagonal for the weight generators, strictly upper triangular for the raising generators and, finally, strictly lower triangular for lowering generators.

Consider, for example, the triplet state of a six-electron system $(S = 1, N = 6)$ characterized by the irrep $(4,2)$ of $U(2)$. The three components of this triplet are labeled by the standard Weyl tableaux, shown in Fig. 4a. Eliminating the boxes carrying an index 2, we obtain from them the $U(1)$ Weyl tableaux, shown in Fig. 4b. Designating the irrep $(4,2)$ subduced by $U(1)$ as $(4,2) \downarrow U(1)$,[22] we get

$$(4,2) \downarrow U(1) = (4) + (3) + (2) \quad . \qquad (60)$$

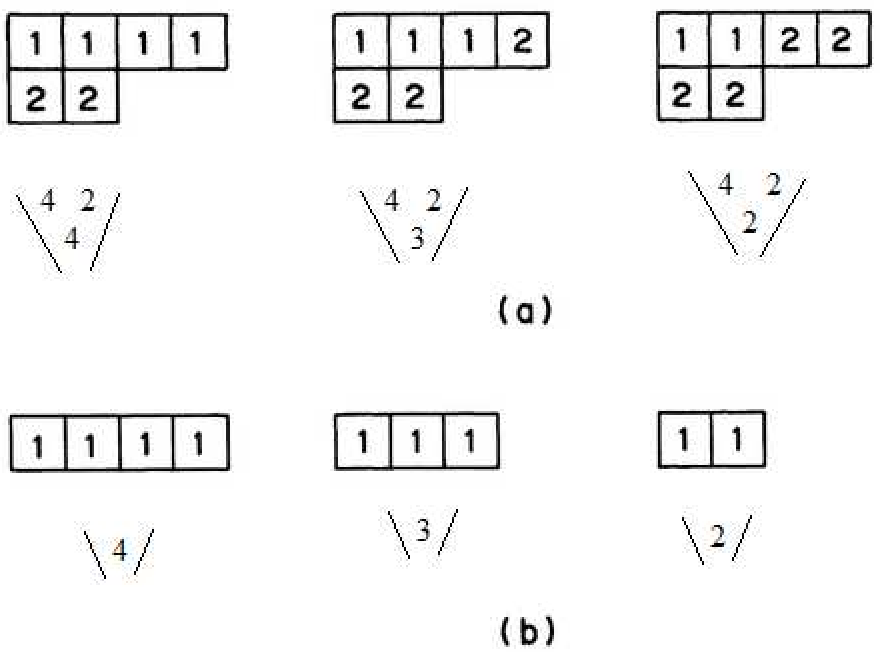
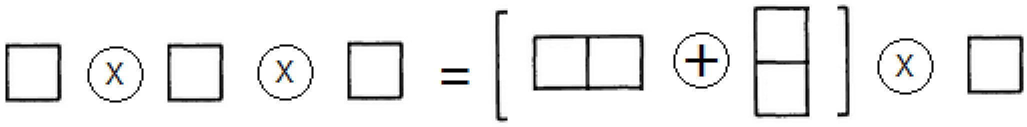


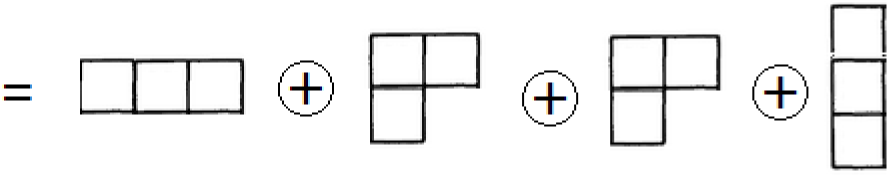
Fig. 4
Example of subduction of $U(2)$ states to $U(1)$.

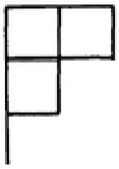
We note that these irreps are exactly those characterized by conditions (59).

Since $U(1)$ has only one-dimensional irreps, this subduction labels uniquely all the basis vectors of a given $U(2)$ irrep carrier space. This idea has been exploited in the general case of an arbitrary $U(n)$ irrep by Gelfand and Tsetlin[23], who employed the chain

$$U(n) \supset U(n-1) \supset \ldots \supset U(2) \supset U(1) \quad . \qquad (61)$$





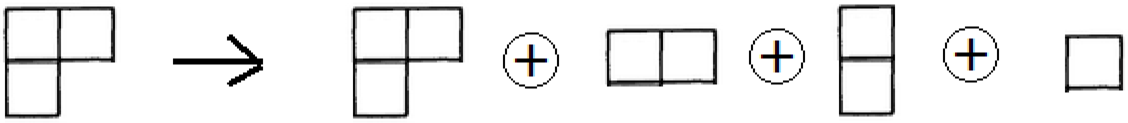
SU(3) N=3 e.g. octuplet  [2 1 0]

2 1 0	2 1 0	2 1 0	2 1 0	2 1 0	2 1 0	2 1 0	2 1 0
2 1	2 1	2 0	2 0	2 0	1 1	1 1	1 0
2	1	2	1	0	1	0	1

Subdueix en SU(2) un triplet [2 0] dos doblets [2 1], [1 1] i un singulet [1 0]:

$$\begin{matrix}2\,1 & 2\,1 & 2\,0 & 2\,0 & 2\,0 & 1\,1 & 1\,1 & 1\,0\end{matrix}$$

$$\underbrace{2 \qquad 1}_{\text{doblet}} \quad \underbrace{2 \qquad 1 \qquad 0}_{\text{triplet}} \quad \underbrace{1 \qquad 0}_{\text{doblet}} \quad \underbrace{1}_{\text{singulet}}$$



$\begin{matrix}3 & 4\\ 2 & \end{matrix} \Big/ \begin{matrix}3 & 1\\ 1 & \end{matrix} = 8$		$\begin{matrix}2 & 3\\ 1 & \end{matrix} \Big/ \begin{matrix}3 & 1\\ 1 & \end{matrix} = 2$	2 3 / 2 1 = 3	$\begin{matrix}2\\ 1\end{matrix} \Big/ \begin{matrix}2\\ 1\end{matrix} = 1$	2 / 1 = 2
octuplet	--->	doblet	triplet	singulet	doblet