Prove this theorem: *In one dimension there are no degenerate bound states.*

Hint:
Suppose there are *two* solutions, ψ_1 and ψ_2, with the same energy E. Multiply the Schrödinger equation for ψ_1 by ψ_2, and the Schrödinger equation for ψ_2 by ψ_1, and subtract, to show that $(\psi_2 d\psi_1/dx - \psi_1 d\psi_2/dx)$ is a constant. Use the fact that for normalizable solutions $\psi \to 0$ at $\pm\infty$ to demonstrate that this constant is in fact zero. Conclude that ψ_2 is a multiple of ψ_1, and hence that the two solutions are not distinct.

$$\left.\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + V\psi_1 = E\psi_1 &\Rightarrow -\frac{\hbar^2}{2m}\psi_2\frac{d^2\psi_1}{dx^2} + V\psi_1\psi_2 = E\psi_1\psi_2 \\ -\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2} + V\psi_2 = E\psi_2 &\Rightarrow -\frac{\hbar^2}{2m}\psi_1\frac{d^2\psi_2}{dx^2} + V\psi_1\psi_2 = E\psi_1\psi_2 \end{aligned}\right\} \Rightarrow -\frac{\hbar^2}{2m}\left[\psi_2\frac{d^2\psi_1}{dx^2} - \psi_1\frac{d^2\psi_2}{dx^2}\right] = 0.$$

But $\frac{d}{dx}\left[\psi_2\frac{d\psi_1}{dx} - \psi_1\frac{d\psi_2}{dx}\right] = \frac{d\psi_2}{dx}\frac{d\psi_1}{dx} + \psi_2\frac{d^2\psi_1}{dx^2} - \frac{d\psi_1}{dx}\frac{d\psi_2}{dx} - \psi_1\frac{d^2\psi_2}{dx^2} = \psi_2\frac{d^2\psi_1}{dx^2} - \psi_1\frac{d^2\psi_2}{dx^2}.$

Since this is zero, it follows that $\psi_2\frac{d\psi_1}{dx} - \psi_1\frac{d\psi_2}{dx} = K$ (a constant).

But $\psi \to 0$ at ∞ so the constant must be zero. Thus $\psi_2\frac{d\psi_1}{dx} = \psi_1\frac{d\psi_2}{dx}$, or $\frac{1}{\psi_1}\frac{d\psi_1}{dx} = \frac{1}{\psi_2}\frac{d\psi_2}{dx}$,

so $\ln\psi_1 = \ln\psi_2 +$ constant, or $\psi_1 = (\text{constant})\psi_2$.