for every function, $f \in C^\infty(M)$, the Laplacian, Δf, is given by

$$\Delta f = \frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(\sqrt{|g|}\, g^{ij} \frac{\partial f}{\partial x_j} \right).$$

for $M = \mathbb{R}^n$ with its standard coordinates, the Laplacian is given

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}.$$

Laplacian in spherical coordinates,

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \qquad (g^{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2} \sin^{-2}\theta \end{pmatrix}$$

$$\sqrt{|g|} = r^2 \sin\theta$$

$$\Delta f = \frac{1}{r^2 \sin\theta} \begin{pmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \end{pmatrix} r^2 \sin\theta \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2} \sin^{-2}\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \varphi} \end{pmatrix}$$

$$\Longrightarrow \quad \Delta f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \varphi^2}.$$

the metric, $(\widetilde{g}_{ij})$, on S^2 is given by the matrix

$$(\widetilde{g}_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \qquad (\widetilde{g}^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-2}\theta \end{pmatrix}.$$

$$\Delta_{S^2} f = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2},$$

it has been shown that there is an effective potential induced by the curvature of the surface.

$$V_S(q_1, q_2) = -\frac{\hbar^2}{2m}(M^2 - K) = -\frac{\hbar^2}{8m}(k_1 - k_2)^2$$

where k_1 and k_2 are the principal curvatures of the surface S and

$$M = \tfrac{1}{2}(k_1 + k_2) \quad \text{(mean curvature)}$$

$$K = k_1 k_2 \quad \text{(Gaussian curvature)}$$

(da Costa R C T 1981 Quantum mechanics of a constrained particle *Phys. Rev.* A 23 1982–7)

This effective potential arises from the separation of variables of the Schrödinger equation formulated for the surface in question using a coordinate system with two coordinates tangential to the surface and one coordinate normal to it. When the normal coordinate is separated from the equation, the effective potential V_S remains.

in spherical coordinates $k_1 = k_2 \Longrightarrow V_S(q_1, q_2) = 0$

(not in general)