for every function, $f \in C^\infty(M)$, the Laplacian, $\Delta f$, is given by

$$\Delta f = \frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial x_i} \left( \sqrt{|g|} g^{ij} \frac{\partial f}{\partial x_j} \right).$$

for $M = \mathbb{R}^n$ with its standard coordinates, the Laplacian is given

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}.$$

Laplacian in spherical coordinates,

$$\begin{pmatrix} g_{ij} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \quad \quad \begin{pmatrix} g^{ij} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2} \sin^{-2} \theta \end{pmatrix}$$

$$\sqrt{|g|} = r^2 \sin \theta$$

$$\Delta f = \frac{1}{r^2 \sin \theta} \left( \frac{\partial}{\partial r} \begin{pmatrix} \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \end{pmatrix} \right) r^2 \sin \theta \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2} \sin^{-2} \theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \varphi} \end{pmatrix}$$

$$\implies \Delta f = \frac{1}{r^2 \partial_r} \left( r^2 \partial_r f \right) + \frac{1}{r^2 \sin \theta \partial \theta} \left( \sin \theta \partial_\theta f \right) + \frac{1}{r^2 \sin^2 \theta \partial \varphi^2} \partial_\varphi^2 f.$$
the metric, \((\widetilde{g_{ij}})\), on \(S^2\) is given by the matrix

\[
(\widetilde{g_{ij}}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}, \quad (\widetilde{g^{ij}}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-2} \theta \end{pmatrix}.
\]

\[
\Delta_{S^2} f = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2},
\]

it has been shown that there is an effective potential induced by the curvature of the surface.

\[
V_s(q_1, q_2) = -\frac{\hbar^2}{2m} (M^2 - K) = -\frac{\hbar^2}{8m} (k_1 - k_2)^2
\]

where \(k_1\) and \(k_2\) are the principal curvatures of the surface \(S\) and

\[
M = \frac{1}{2}(k_1 + k_2) \quad \text{(mean curvature)}
\]

\[
K = k_1 k_2 \quad \text{(Gaussian curvature)}
\]


This effective potential arises from the separation of variables of the Schrödinger equation formulated for the surface in question using a coordinate system with two coordinates tangential to the surface and one coordinate normal to it. When the normal coordinate is separated from the equation, the effective potential \(V_s\) remains.

in spherical coordinates \(k_1 = k_2 \implies V_s(q_1, q_2) = 0\)

(not in general)