RECENT RESULTS AND OPEN QUESTIONS RELATING CHU DUALITY AND BOHR COMPACTIFICATIONS OF LOCALLY COMPACT GROUPS

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Abstract. This is an overview of recent results and open questions related to Chu duality and Bohr compactifications, two notions of special topological significance that help to describe a locally compact group by means of its finite dimensional unitary representations.

1. Introduction

In this paper we collect some problems that have appeared in the context of harmonic analysis on locally compact groups but can be understood, and perhaps solved, adopting topological methods. Naturally, this will also produce some genuine topological questions that can be handled using methods of harmonic analysis. We start with a simple example that illustrates quite well the interplay between the two subjects. Consider the group $\mathbb{Z}$ of integers and let us agree to say that a sequence $(n_k) \in \mathbb{Z}$ converges to n_0 when the sequence (t^{n_k}) converges to t^{n_0} for all $t \in \mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$. Are there convergent sequences under this definition?

It may appear that finding some convergent sequence should not be difficult. Suppose however that $\{n_k\}$ is a sequence which goes to 0. Then, by hypothesis, the sequence of functions $\{t^{n_k}\}$ converges pointwise to 1 on $\mathbb{T}$. Or, equivalently, the sequence of functions $\{e^{i2\pi n_k x}\}$ converges pointwise to 1 on the interval $[0, 1]$. Applying Lebesgue's Dominated Convergence Theorem, it follows that the sequence $\{0\} = \{\int_0^1 e^{i2\pi n_k x} dx\}$ converges to $\int_0^1 dx = 1$, which is a contradiction.

Quite surprisingly we have seen that the definition of convergence given above on $\mathbb{Z}$ produces no nontrivial convergent sequences. This convergence actually stems from the initial topology generated by the functions $n \longrightarrow t^n$ of $\mathbb{Z}$ into $\mathbb{T}$. It is called the *Bohr topology* of $\mathbb{Z}$ (denoted $\mathbb{Z}^\sharp$) and is the largest precompact (and, therefore, nondiscrete) group topology that can be defined on the integers. Even though this topology has been widely studied recently, we are still far from understanding it well in general.

There are other *more topological* approaches to show the absence of nontrivial convergent sequences in $\mathbb{Z}^\sharp$. The one we shall focus on in this paper is based on a careful study of the mappings $n \longrightarrow t^n$ of $\mathbb{Z}$ into $\mathbb{T}$. When a sequence of integers $\{m_j\}$ is lacunary, i.e. $\frac{m_{j+1}}{m_j} > q > 1$, the subset $A = \{m_j : j \in \mathbb{N}\}$ lives in $\mathbb{Z}^\sharp$ as an *interpolation subset*: that is to say, every real-valued bounded function (regardless of its continuity)

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defined on A can be extended to a continuous function $\overline{f}$ of $\mathbb{Z}^\sharp$ into $\mathbb{R}$ (alternatively, we can say that the subset $\{m_j : j \in \mathbb{N}\}$ is C^*-*embedded* in $\mathbb{Z}^\sharp$). It is easily verified that a convergent sequence cannot be an interpolation set and, since every sequence contains many lacunary subsequences (and, therefore, interpolation sets), it follows that there are no convergent sequences in $\mathbb{Z}^\sharp$.

This property actually extends to all abelian groups. If G is an abelian group, let us denote by $G^\sharp$, the group G equipped with its maximal precompact group topology and by bG the completion of $G^\sharp$. In [10] van Douwen initiated a detailed analysis of the topological properties of $G^\sharp$ and, in doing so, he disclosed to general topologists a collection of questions that had by then been in consideration in Harmonic Analysis for at least 30 years. He in particular proved the following theorem that we take as our starting point.

Theorem 1.1. [10] *If G is an abelian group, every $A \subset G$ contains a subset D with $|D| = |A|$ that is relatively discrete and C^*-embedded in bG.*

2. Basic Definitions

2.1. **On Chu duality.** Chu duality, called unitary duality by Chu [5], is based on giving a certain topological and algebraic structure to the set of finite dimensional representations of a topological group G. Denote to that end by G^x_n the set of all continuous n-dimensional unitary representations of G. It follows from a result of Goto [19] that the set G^x_n, equipped with the compact-open topology, is a locally compact space. The space $G^x = \sqcup_{n<\omega} G^x_n$ (as a topological sum) is called the *Chu dual* of G [5].

The algebraic structure of G^x is given by two standard operations: the direct sum and the tensor product of representations, that are induced by the corresponding operations between finite dimensional operators.

- $(\pi \oplus \pi')(x) = \pi(x) \oplus \pi'(x)$, for all $\pi, \pi' \in G^x$ and $x \in G$.
- $(\pi \otimes \pi')(x) = \pi(x) \otimes \pi'(x)$, for all $\pi, \pi' \in G^x$ and $x \in G$.

There is also a concept of equivalence for representations that is often useful: two representations $\pi_1, \pi_2 \in G^x_n$ are said to be (unitarily) *equivalent*, in symbols $\pi_1 \sim \pi_2$, when there is a unitary matrix U such that $\pi_1(x) = U^{-1}\pi_2(x)U$ for all $x \in G$. This clearly defines an equivalence relation in G^x.

The main feature of Chu duality is the construction of a *bidual* of G from the Chu dual G^x. Denoting by $\mathcal{U} = \sqcup_{n<\omega}\mathcal{U}(n)$, the topological sum of the spaces $\mathcal{U}(n)$ of $n \times n$ unitary matrices (topologized as subsets of $\mathbb{C}^{n^2}$), this bidual consists of the so-called continuous *quasi-representations*, i.e. mappings $Q : G^x \longrightarrow \mathcal{U}$ satisfying:

- $Q[G^x_n] \subset \mathcal{U}(n)$.
- $Q(\pi \oplus \pi') = Q(\pi) \oplus Q(\pi')$, for all $\pi, \pi' \in G^x$.
- $Q(\pi \otimes \pi') = Q(\pi) \otimes Q(\pi')$, for all $\pi, \pi' \in G^x$.
- $Q(U^{-1}\pi U) = U^{-1}Q(\pi)U$, $\pi \in G^x_n$, $U \in \mathcal{U}(n)$.

See [5] or [33] or [34] for details.

The set of all *continuous* quasi-representations of G equipped with the compact-open topology is a topological group with pointwise multiplication as composition law, called the *Chu quasi-dual group* of G and denoted by G^{xx}. The evaluation map $\epsilon : G \longrightarrow G^{xx}$ establishes a group homomorphism between G and G^{xx} that gives a measure of how strongly finite dimensional representations determine the structure of G. This homomorphism is one-to-one if and only if continuous finite dimensional representations separate points of G. Groups with that property are usually called maximally almost periodic, or MAP for short, and constitute the natural scope of Chu duality. The map ϵ is always continuous on compacta (an application of Ascoli's theorem) and hence ϵ is continuous for every locally compact group . When $\epsilon : G \longrightarrow G^{xx}$ is in addition open and surjective (i.e. it is an isomorphism of topological groups) G is said to *satisfy Chu duality* or to be *Chu reflexive* (or simply *Chu*). Using this terminology, one has [5] that LCA groups and compact groups satisfy Chu duality (Chu duality actually reduces to the dualities of Pontryagin and Tannaka-Kreĭn respectively for such groups). There is a duality theory for non abelian groups which is based on infinite-dimensional representations (a recent account of duality theory of locally compact groups is given in [12]). We shall not touch on this duality here.

2.2. **The Bohr compactification.** The Bohr compactification of a topological group G, can be defined as a pair (bG, b) where bG is a compact Hausdorff group and b is a continuous homomorphism from G onto a dense subgroup of bG such that every continuous homomorphism $h\colon G \to K$ into a compact group K extends to a continuous homomorphism $h^b\colon bG \to K$, making the lower triangle in the following diagram commutative:

$$\begin{tikzcd}
 & G^{xx} \arrow[dr, "j"] & \\
G \arrow[ur, "\epsilon"] \arrow[rr, "b"] \arrow[dr, "h"'] & & bG \arrow[dl, "h^b"] \\
 & K &
\end{tikzcd}$$

The upper triangle of this diagram gives the relation between Bohr compactifications and Chu duality. Chu [5] proved that the group of all quasi-representations of G (*continuous or not*), equipped with the topology of pointwise convergence on G^x provides a realization of bG. As G^{xx} consists of *continuous* quasi-representations, the inclusion homomorphism $j\colon G^{xx} \to bG$ that appears in the above diagram is clearly continuous and one-to-one.

The topology that b induces on G, will be referred to as the *Bohr topology.* Since $b = j \circ \epsilon$, the map b will be one-to-one exactly when ϵ is, in other words, the Bohr topology will be Hausdorff precisely when G is MAP. Since compact groups (and, in particular, bG) are completely determined by their finite-dimensional representations (this is Tannaka-Kreĭn duality), the Bohr topology of a group G may also be defined as the one that G inherits from its embedding in the product $\mathcal{U}^{G^x}$. We refer to [33, V, §14] or to [34] for a careful examination of bG and its properties.

3. Abelian Groups

In the case of Abelian groups, the notions introduced above become essentially simpler. This is due to the fact that, for Abelian groups, all irreducible representations are one dimensional; that is, homomorphisms into the *torus*, $\mathbb{T}$, the group of all complex numbers of modulus one. These one dimensional representations are called *characters* and are the building blocks of the duality theory of Abelian groups (see [55]).

Let (G, τ) be an arbitrary topological abelian group. A *character* on (G, τ) is a continuous homomorphism χ from G to the torus $\mathbb{T}$. The set $\widehat{G}$ of all characters, equipped with the compact open topology, is a topological group with pointwise multiplication as the composition law, which is called the *dual group* of (G, τ). There is a natural evaluation homomorphism $\epsilon\colon G \longrightarrow \widehat{\widehat{G}}$ of G into its bidual group. We say that a topological abelian group (G, τ) satisfies *Pontryagin-van Kampen duality* if the evaluation map ϵ is a topological isomorphism onto. The Pontryagin-van Kampen theorems establish that every LCA group satisfies P-vK duality.

In [10] van Douwen proved, among other things, the remarkable Theorem 1.1. Except for the standing abelian hypothesis, his proofs of results concerning $\sharp$-groups made no use whatsoever of specific algebraic properties. This probably led him to ask whether two groups G_1 and G_2 with the same cardinality should have $G_1^\sharp$ and $G_2^\sharp$ homeomorphic. Some years later Kunen [36] and, independently, Dikranjan and Watson [9], gave examples of *torsion* groups with the same cardinality having non-homeomorphic $\sharp$-spaces. Still, much remains unknown. Actually, it is not yet known what happens with some utterly elementary groups:

Question 1. *Are the spaces $\mathbb{Z}^\sharp$ and $(\mathbb{Z}\times\mathbb{Z})^\sharp$ are homeomorphic? What about the spaces $\mathbb{Q}^\sharp$ and $\mathbb{Z}^\sharp$?*

One consequence of Theorem 1.1 is that $\sharp$-groups cannot contain infinite compact subsets. Indeed, the closure $\mathrm{cl}_{bG}D$ of a discrete and C^*-embedded subset D of bG is homeomorphic to βD, the Stone-Čech compactification of the *discrete* space D. If A is a compact subset of $G^\sharp$ and $D \subset A$ is as in Theorem 1.1, we obviously have $\mathrm{cl}_{bG}D \subset A$. But $|\mathrm{cl}_{bG}D| = |\beta D| = 2^{2^{|D|}} = 2^{2^{|A|}}$. This is a particular case of a general fact true for any LCA group, a pivotal result indeed about the Bohr topology of LCA groups.

Theorem 3.1 (Glicksberg, 1962 [18])**.** *Let G be an LCA group. If $A \subset G$ is compact in bG, then A is compact in G.*

Theorem 3.1 in its full generality can actually be deduced from Theorem 1.1 and, conversely, Theorem 1.1 follows from Theorem 3.1, by way of Rosenthal's ℓ^1 theorem. These relations will be explored in Section 4. Further properties concerning the Bohr topology of a LCA group can be found in [6, 7, 16, 31]

4. Nonabelian groups

Here we will focus on determining to what extent the results concerning duality theory and Bohr topology of abelian groups can be extended to the noncommutative context. The first contributions to these program have been given by Chu [5], Heyer [33, 34], Landstad [38], Moran [43], Poguntke [46, 48, 47], and Roeder [53]. Recent contributions to the subject can be found in [8, 17, 24, 25, 26, 30, 51, 52, 62]. Nevertheless, we do not know how Chu reflexive groups are placed within the class of LC groups and this is one of the major difficulties for understanding Chu duality. Therefore, the main question here is:

Question 2. *Characterize the (necessarily MAP) locally compact groups that satisfy Chu duality.*

Obviously Q2 leaves open a good number of other questions about Chu duality. Firstly, we give a brief account on the subject. A topological space X is called *hemicompact* if there is a countable family of compact subsets $(K_n)_n$ of G such that every compact subset L of G is contained in some K_n.

Proposition 4.1. *Let G be a locally compact MAP group.*

1. *If G is discrete (resp. metrizable), then G_n^x is compact (resp. hemicompact).*
2. *Conversely, if G is compact then each equivalence class defined by $\sim$ is open. Therefore, the quotient space $\frac{G^x}{\sim}$ is discrete.*

(3) *If G is second countable then G^x_n and G^{xx} are second countable. As a consequence G^x is metrizable. In this case G is Chu-reflexive if and only if the evaluation map ϵ is onto.*
(4) *G^{xx} need not be locally compact, even for countable G,* [30, 52].

Now, we recall a notion due to Takahashi [58] in order to obtain a representation of the Chu quasi-dual for some classes of groups. For each locally compact group G, Takahashi has constructed a locally compact group G_T called *Takahashi quasi dual* such that G_T is maximally almost periodic, and G'_T is compact. The category of locally compact groups with these two properties is denoted by TAK . If $n > 1$ and $D \in Hom_c(G, U(n))$ then the sets $t_n(D; U) = \{D \otimes \chi : \chi \in U\}$, U any neighborhood of the identity in the group G^x_1, form a fundamental system of neighbourhoods of D for a topology in $Hom_c(G, U(n))$. We denote by G^t_n the set $Hom_c(G, U(n))$ equipped with this topology and the symbol G^t denotes the topological sum of the spaces G^t_n, for $n \in \mathbb{N}$. A *unitary mapping* on G^t is a continuous mapping $p : G^t \longrightarrow \mathcal{U}$ conserving the main operations between unitary representations (see [46] for details). The set of all unitary mappings on G^t equipped with the compact-open topology is a topological group, with pointwise multiplication as the composition law, which is usually called the *Takahashi quasi-dual group* of G and is denoted by G_T. It is easily verified that $G^{xx} \subset G_T \subset bG$. The Takahashi duality theorem establishes that $G \cong G_T$ if $G \in$ TAK . A detailed discussion and extension of this theory has been given by Poguntke in [46].

Concerning Chu duality, the first difficulty is to identify the quasidual group G^{xx} of a locally compact group G. Some extreme situations, totally alien to the abelian case may actually appear, as for instance that $G^{xx} = bG$ or, what is the same (see [17, 30]), that $\widehat{G}_n$ is discrete for every n. Next follows some examples that illustrate the different situations that may arise.

Example 4.2. *(Moran* [43]*) Let $\{p_i\}$ be an infinite sequence of distinct prime numbers ($p_i > 2$), and let F_i be the projective special linear group of dimension two over the Galois field $GF(p_i)$ of order p_i. If $G = \sum_{i\in\mathbb{N}} F_i$, we have $G^{xx} = G_T = bG$.*

Example 4.3. *(Heyer* [34]*) Let $\mathbb{Z}_3 \rtimes \mathbb{Z}_2 = S_3$ the permutation group. Define $G_i = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ for all $i \in \mathbb{N}$ and take $G = \sum_{i\in\mathbb{N}} G_i$. Then $G = G^{xx}$ and $G_T = \prod_{i\in\mathbb{N}} \mathbb{Z}_3 \rtimes \sum_{i\in\mathbb{N}} \mathbb{Z}_2$.*

Example 4.4. *Let p a prime number greater than 2, and let F_i be the projective special linear group of dimension two over the Galois field $GF(p)$ of order p . If $G = \sum_{i\in\mathbb{N}} F_i$, we have $G^{xx} = G$ and $G_T = bG$.*

More recently, we have the following results (cf. [30]).

Proposition 4.5. *Let G be a simple MAP discrete group (which implies $G' = G$). Then the following conditions are equivalent:*
(i) *$G^{xx} = G_T$;*
(ii) *$\widehat{G}_n$ is discrete for all $n \in \mathbb{N}$;*
(iii) *$G^{xx} = bG$.*

Proposition 4.6. *Let G be a discrete MAP group that is nilpotent of length two, and such that for each positive integer n there are only finitely many co-finite normal subgroups H of G' whose index is less or equal than n. Then $G^{xx} \cong G_T$.*

Proposition 4.6 is a variation of the following nice result due to Poguntke. [48, 47].

Corollary 4.7. *(Poguntke, 1976) The Heisenberg integral group H, satisfies that $H^{xx} \cong H_T$.*

Theorem 4.8. *Let G be a discrete MAP group that is an FC group and, for each positive integer n, there are only finitely many co-finite normal subgroups H of G' such that G'/H accepts faithful representations into $U(n)$. Then $G^{xx} \cong G_T$.*

Corollary 4.9. *Let $G = \sum_{n\in\mathbb{N}} F_n$, where each F_n' is simple and $lim_{n\to\infty} exp(F_n') = \infty$. Then $G^{xx} \cong G_T$.*

Furthermore, for an FC group we have.

Theorem 4.10. *Let G be an FC group and suppose there is $N \in \mathbb{N}$ such that $exp(G') \leq N$ and $mdus(G/H) \leq N$ for all normal subgroup H of G that is co-finite in G'. Then the group G is Chu reflexive.*

Examples 4.3 and 4.2 also follow from Theorem 4.10. Finally, next example shows that the Chu quasi-dual group G^{xx} need not be locally compact even for a countable discrete group G.

Example 4.11. *Let $\{p_n\}$ be an infinite sequence of distinct prime numbers ($p_n > 2$), and let $G_n = PSL(2, p_n)$ be the projective special group of dimension two over the finite filed of order p_n. For each n, let $G_{n,m}$ be a copy of G_n, for $m = 1, 2, \ldots$. Let $G = \sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} G_{n,m})$ with the discrete topology. The group G^{xx} is not locally compact.*

We have already mentioned the most recent results about the unitary or Chu duality (cf. [17, 30, 52]). Nevertheless, the subject is far from being settled. There are already too many open question that obstruct the progress in this are of research.

The following two questions concern the very basic structure of Chu duality, their solution would be of much importance for the study of Chu duality.

Question 3 (Poguntke, 1976 [48])**.** *Let G be a locally compact MAP group with evaluation map $\epsilon\colon G \to G^{xx}$. Is $\epsilon(G)$ dense in G^{xx}?*

Chu [5] asserted that $bG = bG^{xx}$ for every locally compact group G. The proof in [5] of this fact is however incomplete, and this remains indeed as one of the main open questions.

Question 4 (Wu, 2000)**.** *Let G be a locally compact MAP group. Is it true that $bG = bG^{xx}$?*

A positive solution to Q3 would imply a positive solution to Q4.

Question 5 (Chu, 1966 [5])**.** *Does the free group with two generators, equipped with the discrete topology, satisfy Chu duality?*

This question appears as one of the major difficulties for a full understanding of Chu duality. Avoiding it does not however answer all questions.

Question 6. *Characterize the MAP, discrete groups without free non abelian subgroups that satisfy Chu duality.*

Among groups with no free (non abelian) subgroups, amenable groups are especially important. A topological group G is *amenable* when the Banach space $\ell^\infty(G)$ admits a left-invariant mean, that is, a continuous linear functional Λ on $\ell^\infty(G)$ with $\Lambda(1) = 1$ and $\Lambda(L_x f) = \Lambda(f)$, for every $x \in G$ and every $f \in \ell^\infty(G)$ (here L_x denotes as usual the left action of x on f, $(L_x f)(g) = f(x^{-1}g)$). For discrete G, this is equivalent, to the existence of a finitely-additive left-invariant probability measure on G. Amenability has a strong impact on the representation-theoretic properties of a locally compact group, see [44] for instance. Compact and abelian groups are amenable while any group having a discrete free nonabelian subgroup is not.

Question 7. *Characterize the MAP amenable, locally compact groups that satisfy Chu duality.*

5. How is G placed in bG? Interpolation sets

Definition 5.1. *Let G be a topological group and let X denote a point-separating, uniformly closed, self-adjoint subalgebra of $CB(G)$ (continuous, complex-valued, bounded functions on G). A subset S of G is said to be an X-interpolation set provided that every bounded function $f\colon S \to \mathbb{C}$ (continuous or not) can be extended to a function $\bar{f}\colon G \to \mathbb{C}$ with $\bar{f} \in X$.*

Each closed subalgebra X as above is a commutative C^*-algebra and we can apply the full-strength of Gelfand's duality to it, see for instance [55]. Let $\sigma(X)$ denote the space of multiplicative linear functionals on X (i.e. linear functionals $T\colon X \to \mathbb{C}$ with $T(fg) = T(f)T(g)$, for all $f, g \in X$). The set $\sigma(X)$ with the topology of pointwise convergence on X is a compact topological space called the *spectrum* of X. Every element $f \in X$ can then be identified with a function $E_f \in C(\sigma(X), \mathbb{C})$ via evaluations ($E_f(T) = T(f)$ for every $T \in \sigma(X)$). The main consequence of Gelfand's duality is that this identification establishes an isomorphism of C^*-algebras.

The compact space $\sigma(X)$ also defines a compactification of G. Taking into account that the elements of X are continuous functions on G, we have an evaluation mapping $j\colon G \to \sigma(X)$ (given by $j(g)(T) = T(g)$) that defines a one-to-one continuous map with dense range. From this point of view X-interpolation sets are those subsets of G that are discrete and C^*-embedded in $\sigma(X)$.

The Bohr compactification can be obtained in the preceding way by considering $X = AP(G)$, the algebra of almost periodic functions on G. A bounded function $f\colon G \to \mathbb{C}$ is *almost periodic* if the set of translates $\{L_x f\colon x \in G\}$ is a compact subset of $CB(G)$ (for the topology of uniform convergence). A function $f\colon G \to \mathbb{C}$ turns to be almost periodic if and only if it is the uniform limit of matrix coefficients[1] of finite-dimensional unitary representations. Thus the almost periodic functions are precisely the functions that admit a continuous extension to bG. The spectrum $\sigma(AP(G))$ of $AP(G)$ can then be identified with the Bohr compactification of G.

The X-interpolation sets for $X = AP(G)$ are called I_0-sets. I_0-sets were first studied by Hartman and Ryll-Nardzewski in the sixties in a series of papers starting with [27, 28]. The fact that lacunary sequences of integers are I_0-sets was first proved in [57] (see [37] for a recent proof).

5.1. Existence and abundance of interpolation sets.

Existence problems on interpolation sets are amenable to topological techniques as the proof of van Douwen's theorem 1.1 [10] shows (see [15] for a simpler proof). We recast here Theorem 1.1 in terms of I_0-sets:

Theorem 5.2. *Every infinite subset A of a discrete abelian group G contains an I_0-set S with $|S| = |A|$.*

A sequence $S = (x_n)_n$ of a Banach space E is said to be an (or equivalent to the) ℓ^1-basis if the map sending x_n to the canonical basis (e_n) of ℓ^1 extends to a

[1] If π is a unitary representation of a group G on a Hilbert space $\mathbb{H}$, a matrix coefficient of π is a complex-valued function $g \mapsto \langle \pi(g)\xi, \eta \rangle$, with $\xi, \eta \in \mathbb{H}$.

linear homeomorphism on the closed linear span of S. Interpolation sets share many properties with ℓ^1-basis (see for instance [3], and what follows). This relation can be a very fruitful one, mainly because the existence of ℓ^1-basis is neatly characterized by Rosenthal's well-known theorem.

Theorem 5.3 (Rosenthal 1971, [54]). *A bounded sequence in a Banach space either has a weakly Cauchy subsequence or has a subsequence which is an ℓ^1-basis.*

If E is a Banach space, a sequence (x_n) in E is a *weakly Cauchy* sequence if $f(x_n)$ is convergent for every continuous linear functional f on E.

Rosenthal's theorem relates the presence of ℓ^1-basis to the absence of weakly convergent sequences. It can be adapted to provide a similar relation with interpolation sets, this is the Rosenthal-type theorem for locally compact groups that appears in [17].

Theorem 5.4. *Let G be a metrizable locally compact group. A sequence in G admitting no Bohr Cauchy subsequence (i.e. no subsequence converging to an element of bG), must contain an infinite I_0-subset.*

Observe that the combination of Glicksberg's Theorem 3.1 and Theorem 5.4 implies Theorem 1.1 for countable abelian groups. As indicated in Section 2, it is also true that Theorem 3.1 follows from an appropriate extension of Theorem 1.1 to nondiscrete groups, see [16], a fact that was used there to prove Glicksberg's-type theorems for some abelian nonlocally compact groups. These theorems are mainly based on the following analog of Theorem 1.1 that appears in [16]:

Theorem 5.5. *Let G be abelian, locally connected and Čech-complete. Every subset A of $\widehat{G}$ that is not equicontinuous as a set of $\mathbb{T}$-valued functions on G, must contain an infinite I_0-set.*

Both approaches have failed so far to provide a general answer for the simplest questions about I_0 sets in the case of nonabelian locally compact (even discrete) groups. The relevant question here therefore is:

Question 8. *Which (countable) discrete groups contain no nontrivial Bohr convergent sequences? Or equivalently, which groups G have infinite I_0-sets inside every infinite subset $A \subset G$?*

As far as we know the first noncommutative theorem related to Q8 was given by Moran [43]. We need the concept of direct integral of a representation to understand his result. Roughly, the direct integral $\pi = \int_A^{\oplus} \pi_\alpha \, d\mu(\alpha)$ of a family of unitary representations π_α, where $\alpha \in A$ runs on a measure space (A, μ), is another representation such that its matrix coefficients are obtained as ordinary integrals, of the matrix coefficients of the representations π_α.

Theorem 5.6 (Moran, 1971 [43]). *Let G be a locally compact group and suppose its left regular representation can be decomposed as a direct integral of representations* almost all *of which are finite-dimensional. Then every Bohr convergent sequence of G is also convergent in the locally compact topology.*

Corollary 5.7. *Let G be a group that satisfies the hypothesis of the theorem above. Then every subset A of G either has compact closure or contains an infinite I_0-set.*

It happens that *every* unitary representation of a locally compact group may be obtained (often in several unrelated ways) as a direct integral of irreducible representations. With this fact in mind, Theorem 5.6 applies directly to those groups G

whose irreducible representations are all finite dimensional, so-called *Moore groups*. In this line, Remus and Trigos-Arrieta have proved the following result that avoids direct integrals.

Theorem 5.8 (Remus and Trigos-Arrieta, 1999 [51])**.** *If the locally compact group G is Moore then G respects compactness.*

In the opposite direction we have.

Theorem 5.9 (Wu and Riggins, 1996 [62])**.** *Let G be a maximally almost periodic FC group that contains no nontrivial convergent sequences. Then G is abelian by finite (that is has a normal Abelian subgroup of finite index).*

These results leave open the following main question.

Question 9. *Let G be a discrete group that contains no nontrivial convergent sequences.*

(a) *Is G abelian by finite?*
(b) *Can the left regular representation be decomposed as direct integral of finite dimensional representations?*

In the positive direction, we have the following result that appears in [29].

Theorem 5.10. *Let G be a finitely generated discrete group without non-abelian free subgroups. Then G has no non-trivial Bohr convergent sequences if and only if G is abelian by finite.*

Theorem 5.10 displays some examples of discrete groups that have Bohr convergent sequences despite having some good commutativity properties. For instance, the Heisenberg integral group, the *lamplighter group* $(\sum_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}$ or the direct sum $G = \sum_{n\in\mathbb{N}} F_n$, with F_n finite, simple and non-abelian. In order to solve Q8, one has to overcome an important obstacle, namely that of dealing with the free group with two generators.

Question 10. *Does $F(a,b)$ contain non-trivial Bohr convergent sequences?*

The question may be extended to:

Question 11. *Characterize the MAP locally compact groups whose compact subsets are the same in the original and Bohr topologies.*

In connection with the Bohr topology of locally compact groups, several authors have considered the so-called *van der Waerden* (or *self-bohrifying*) groups. That is, compact groups G satisfying that $bG_d = G$, where G_d denotes the same algebraic group G equipped with the discrete topology. In this direction van der Waerden proved that every (algebraic) homomorphism from a compact connected semisimple Lie group into a compact group is continuous (cf. [61]). We mention here the following question along this direction, see [25].

Question 12. *What are the direct products $G = \prod_{i\in I} F_i$ of finite groups F_i such that the Bohr compactification of G_d is topologically isomorphic to G?*

5.2. **Sidon sets.** The matrix coefficients of all (finite- or infinite-dimensional) unitary representations also constitute an algebra. This is the Fourier-Stieltjes algebra B(G) introduced by Eymard in [13]. When G is abelian B(G) reduces to the set of Fourier-Stieltjes transforms of measures of the dual group, see [11].

Let $\overline{\mathrm{B(G)}}$ denote the uniform closure of the Fourier-Stieltjes algebra of G. $\overline{\mathrm{B(G)}}$-interpolation sets in discrete (and locally compact) abelian groups G have been deeply studied under the name of *Sidon* sets, see for instance [40]. It should be remarked that $\overline{\mathrm{B(G)}}$-interpolation sets on noncommutative groups also appear in the literature as *weak Sidon* sets, see [45] for instance.

With these definitions in mind, one has that Sidon subsets of a group G are discrete and C^*-embedded in the spectrum of $\overline{\mathrm{B(G)}}$. Following [41] we will refer to this spectrum as the *Eberlein compactification* of G and denote it by eG. There are two main differences between bG and eG. Firstly, eG is no longer a topological group, only a semitopological semigroup, secondly eG is a proper compactification of G, the embedding of G in eG is a homeomorphism and thus a Glicksberg's-type theorem makes no sense for eG. The question on which sequences contained in G converge to some point in eG (we will refer to this property as being *eG-Cauchy*) does however make sense, and is the truly relevant one. After adapting Rosenthal's theorem to Sidon sets (as it was done in Theorem 5.4 to I_0-sets) the question that corresponds to Q8 is:

Question 13. *Can discrete groups contain nontrivial eG-Cauchy sequences? or equivalently, does every infinite subset of a discrete group G have infinite Sidon subsets?*

Sidon sets are far more abundant than I_0-sets in noncommutative groups. We have for instance the following counterpart to Theorem 5.9.

Theorem 5.11 (de Michele and Soardi [42])**.** *Any infinite subset of a discrete FC-group contains an infinite Sidon subset.*

It should be noticed, and this goes in the same direction of the preceding theorem, that, contrarily to the I_0-case, Sidon subsets of subgroups of a discrete group G are necessarily Sidon subsets of G. Using this fact, it is easy to see that infinite subsets of solvable groups always contain infinite Sidon sets.The following questions should by the same reason be far easier than Q8 or Q10:

Question 14. *Does every infinite subset of the free group on two generators contain an infinite Sidon subset?*

Question 15. *Does every infinite subset of an amenable discrete group contain an infinite Sidon subset?*

Finally, the main question here is

Question 16 (López and Ross, 1975 [40])**.** *Does every discrete group contain some infinite Sidon set?*

It is important to remark that a discrete group with no infinite Sidon sets must necessarily be a torsion group. Even more, no subgroup of a torsion group may contain infinite Sidon sets at all since Sidon subsets of subgroups of *discrete* groups are Sidon.

The weakness of the algebraic structure of eG is also important for the very existence of Sidon sets. Note that the mere existence of an interpolation set implies that the cardinality of the space eG be at least $2^{\mathfrak{c}}$. Thence the interest on knowing which groups admit some interpolation set and which have none at all. Thanks to the Bourgain-Fremlin-Talagrand theorem, that question has a satisfactory answer for I_0-sets:

Theorem 5.12 ([17])**.** *Let G be a maximally almost periodic second countable topological group. The following assertions are equivalent.*

(1) *G has no I_0-sets.*

(2) *bG is Rosenthal compact (that is, bG is homeomorphic to a compact subset of $B_1(X)$, the space of all first class Baire functions defined on some Polish space X).*
(3) *The Bohr compactification bG of G is metrizable*
(4) $|bG| = \mathfrak{c}$.
(5) *G has at most countably many inequivalent finite dimensional unitary representations.*

Countable groups always have a continuum of pairwise inequivalent irreducible representations (cf. [1]) and the same is true for every connected second countable locally compact group. Since $\overline{B(G)}$ is made from matrix coefficients of general unitary representations, just as $AP(G)$ is made from matrix coefficients of finite dimensional ones, it could be expected that the arguments leading to Theorem 5.12 also imply that every discrete or second countable connected locally compact group contains an infinite Sidon set. Notice that the absence of Sidon sets in G implies that eG is Rosenthal-compact and thus of cardinality $\mathfrak{c}$. But the failure of eG to be a topological group makes statements (3), (4) and (5) of Theorem 5.12 nonequivalent. Take for instance $G = SL(2,\mathbb{R})$, all nonconstant functions in $B(G)$ vanish at infinity, i.e., $B(G) = C_0(G) \oplus \mathbb{C}$, and eG can be identified with the one-point compactification of G. The Eberlein compactification of this group is therefore metrizable despite having uncountably many inequivalent irreducible representations. Concerning the equivalence between (2) and (3) the absence of better examples (in particular of discrete ones) leaves without answer the following question.

Question 17. *Can a non-metrizable Eberlein compactification eG be Rosenthal compact?*

A negative answer would simplify and provide a higher lever of applicability of topological techniques for the questions Q14 - Q16.

5.3. **Other compactifications.** The questions discussed in the preceding subsections regarding AP(G)- and $\overline{\mathrm{B(G)}}$-interpolation sets (i.e. I_0- and Sidon sets, respectively) have easier answers in the case of X-interpolation sets with bigger X. The most immediate case is the algebra of weakly almost periodic functions $X = \mathrm{WAP}(G)$. A bounded function $\phi\colon G \to \mathbb{C}$ is *weakly almost periodic* if the set of translates $\{L_x f : x \in G\}$ is a weakly compact subset of $CB(G)$. Ruppert defines in [56] *translation-finite* sets as those sets $A \subset G$, G discrete, such that every bounded $f\colon G \to \mathbb{C}$ that vanishes off A is weakly almost periodic. These sets, called R_W-sets by Chou [4], are WAP(G)-interpolation sets and by [56, Proposition 13], every infinite subset of a discrete group contains an infinite translation-finite subset. See [14] for an extension of this fact to more general (not necessarily locally compact) topological groups.

5.4. **The structure of Sidon and I_0-sets.** Perhaps the oldest question regarding interpolation sets is whether a Sidon set may be dense in the Bohr compactification. This is open even in the simplest groups:

Question 18. [40, 35] *Can a Sidon subset of $\mathbb{Z}$ be dense in $b\mathbb{Z}$?*[2]

A probabilistic argument due to Katznelson [35] seems to suggest a negative answer for Q18. A theorem of Ramsey [49] shows that Q18 is equivalent to the following one:

[2] A negative answer to the preceding question would leave some space to this one: is the closure of a Sidon subset of G a Helson subset of the Bohr compactification?, see [55] for the definition of a Helson set in a compact group K, needless to say that the whole group is not Helson.

Question 19. *Can a Sidon subset of $\mathbb{Z}$ cluster (in the Bohr topology) at some point of $\mathbb{Z}$?*

Although the equivalence between Questions 18 and 19 could point towards a positive answer to the former, the converse conjecture gains strength if we compare with I_0-sets. By a theorem of Hartman and Ryll-Nardzewski [27] no point of G can be a Bohr-cluster point of an I_0-subset of G (the union of an I_0 set and a point is again I_0 and therefore discrete in bG). It is clear in this regard that a deeper knowledge of the relations between I_0 sets and Sidon sets (see [50] and the references therein) would help with Q18. In particular an affirmative answer to the following question implies a negative to Q18:

Question 20 (Grow, 1987 [21])**.** *Is every Sidon subset of $\mathbb{Z}$ a finite union of I_0-sets?*

Bourgain had already shown in [2] that the answer to Q20 is positive for groups G of bounded order (groups with $mg = 0$ for some $m \in \mathbb{Z}$ and all $g \in G$).

Lefèvre and Rodríguez-Piazza have shown that interpolation sets with a lower degree of lacunarity, namely Rosenthal-sets, can be dense in the Bohr compactification (see [39]). This somehow shows how the case of Sidon sets consitutes a *limiting-case*.

We finally mention two questions that appear in [20] and concern the structure of I_0-sets, just as the preceding questions concern the structure of Sidon sets. We need here the concepts of $I_0(U)$-set and ϵ-Kronecker set. If G is a compact group and $U \subseteq G$, we say $E \subset \hat{G}$ is $I_0(U)$ if every bounded function on E is the restriction of the Fourier-Stieltjes transform of a discrete measure supported on U. A set $E \subset \hat{G}$ is *ϵ-Kronecker* for some $\epsilon > 0$, if for every continuous function $\phi\colon E \to \mathbb{T}$ there exists $x \in G$ such that $|\gamma(x) - \phi(\gamma)| < \epsilon$ for all $\gamma \in E$.

Question 21 (Graham, Hare and Körner, [20])**.** *Is every I_0-set a finite union of sets in a more limited class? Perhaps a finite union of ϵ-Kronecker sets?*

Question 22 (Graham, Hare and Körner, [20])**.** *Is every I_0 set $I_0(U)$ for all $U \subset G$? (this assumes G to be connected) What about when $G = \mathbb{T}$?*

Lack of space has refrained us from referring to another whole lot of problems on interpolation sets that might respond to topological treatment. These concern interpolation sets in dual objects of compact groups, see [22] for a recent account and references to previous results.

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