On Group and Semigroup Compactifications of Topological Groups

Jorge Galindo

Departamento de Matemáticas, Univ. Jaume I, 12071-Castellón, Spain.

E-mail address: jgalindo@mat.uji.es

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Contents

Introduction

These notes have been written as accompanying material for a minicourse to be given in the workshop *Topological Groups: Introduction to Dynamical Systems (*Seminario Internacional Complutense*).*

The main objective of the course has been to put together several directions of research that despite being very close are often introduced and treated separately.

The minicourse consists of 3 lectures of 1 hour each. Clearly enough, most of the material contained in these notes will not be covered in detail, this partly accounts for the irregular level of detail in them. The concept of semigroup compactification is introduced practically from scratch and then we turn to examining the resulting object from several points of view, mostly trying to shed light on the relationship between a group and its compactification.

All our work is directed to the compactifications generated by four algebras of functions: left uniformly continuous functions, weakly almost periodic functions, Fourier-Stieltjes algebra and almost periodic functions. Other algebras such as the algebra of distal functions or measure algebras are not even mentioned. Also we have not considered semigroup compactifications of semigroups even if some portions of what we do here make complete sense (and may be related to important problems) for semigroups. On the other hand, the groups to which our results are addressed are very diverse. At some points we are interested in locally compact nonAbelian groups (as is the case for interpolation sets), at others we point to Banach-Lie groups and other classes of Abelian groups (as for instance in the study of unitary and reflexive representability), while for some other matters discrete Abelian groups already provide questions we cannot answer and have centered our attention (the case of cancellability).

Other elections for both the questions to be treated and the compactifications to be used could have been done, but I have chosen those more closely related to my own work and/or taste in an attempt to reach the minimum level of competence due for the task. We do not have for instance treated other important questions such as semigroup compactifications of large groups and the relation of semigroup compactifications with dynamical systems through enveloping semigroups. Both of these subjects have nonetheless survey papers ([**Pes99**] and [**Pes07a**] for the former and [**Gla07**] for the latter) that provide a considerable insight and make unnecessary further elaboration here.

I have tried to give proper credit to all results that do appear in these notes, hopefully very few will have passed unnoticed. Most of the results have proofs, even if many are only sketches. A good number of these proofs (and some results) have been slightly reworked to fit in the discourse, many others are only sketched and a few have been completely modified. Here I also hope that the number of errors has been kept in a minimum. Unfortunately I have only been able to finish these notes in the eve of the workshop. This means that a proper proofreading has not been performed as it is mandatory in a written text (this *also* accounts in part –see the first paragraph of this introduction– for the irregular level of detail in these notes).

Acknowledgements: I owe gratitude to several people without whom this course would have been impossible to ensamble.

First of all I want to thank Elena Martín-Peinador who invited and encouraged me to give this course. Her efforts in organizing these workshops help to keep together our topological groups community and are truly appreciated.

I would like to thank Salvador Hernández for helping me out in some questions related to uniformities. He is also the author or coauthor of many of the results in chapter 3 that have their roots in my doctoral dissertation, written under his guidance, hence the acknowledgement goes almost without saying.

Mahmoud Filali has taught me most of what I know concerning the algebraic structure of the weakly almost periodic compactification, the material of chapter 4 is modeled on a joint work in progress started in September 2007 that will hopefully be finished in the near future.

I also want to thank Stefano Ferri for discussing and sharing with me his ideas around several issues related to section 2.

Finally I would like to thank M. Megrelishvili for his developing parts of this subject. Even if we have had no contact recently, he has been very helpful in reading, rigourously and constructively, some of my works on the

subject (mainly [**Gal09**] and [**FG**]). It will also be noticed, as I did whilst writing, that at some points I am actually (in general unconsciously but some times with direct references) taking up his point of view– and that maybe it should have been him who writes them.

1

Algebras of functions and Compactifications

1. Some distinguished algebras

These notes are devoted to study the compactifications defined by the sets of functions defined on a topological group G that are introduced in this section. They are all subsets of $\mathcal{CB}(G)$ the algebra of all continuous, bounded functions on G. These algebras are all related to the topological behaviour of translations within these sets, we will use in this regard the symbols L_g and R_g to denote, respectively the left- and right-translation operators

$$L_g(f)(x) = f(gx) \qquad R_g(f)(x) = f(gx) \qquad \text{for } f\colon G \to \mathbb{C} \text{ and } x, g \in G.$$

Definition 1.1. Let G denote a topological group. We introduce here the sets of complex-valued functions that will be our object of attention throughout these notes.

- $\boldsymbol{\mathcal{LUC}}$**(G)**: The set of all *left-uniformly continuous functions on* G, i.e., the set of all functions $f\colon G \to \mathbb{C}$ such that for every $\epsilon > 0$ there is a neighbourhood V of the identity of G such that

$$|f(vg) - f(g)| < \epsilon \quad \text{for all } v \in V \text{ and } g \in G.$$

- $\boldsymbol{\mathcal{WAP}}$**(G)**: The algebra of all *weakly almost periodic functions on* G.

$$\mathcal{WAP}(G) = \left\{ f\colon G \to \mathbb{C}\colon f \begin{array}{l} \text{is continuous and the set } \{L_g f\colon g \in G\} \\ \text{is weakly relatively compact in } CB(G) \end{array} \right\}$$

- $\boldsymbol{\mathcal{B}}$**(G)**: The *uniform closure* of the *Fourier-Stieltjes* algebra $B(G)$, $\mathcal{B}(G) = \overline{B(G)}^{\|\cdot\|_\infty}$. The Fourier-Stieltjes algebra is the linear span (in $\mathbb{C}^G$) of the set of continuous positive-definite functions, where a function $\phi\colon G \to \mathbb{C}$ is said to be *positive-definite* provided

$$\sum_{1\le i,j\le n} \alpha_i \overline{\alpha_j} f(x_i^{-1} x_j) \ge 0,$$

for any $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and $x_1, \dots, x_n \in G$.

- $\boldsymbol{\mathcal{AP}(\mathbf{G})}$: The algebra of all *almost periodic functions on* G.

$$\mathcal{AP}(G) = \left\{ f\colon G \to \mathbb{C}\colon f \begin{array}{c} \text{is continuous and the set } \{R_g f\colon g \in G\} \\ \text{is norm relatively compact in } CB(G) \end{array} \right\}$$

It is well-known that weakly almost periodic functions on locally compact groups are uniformly continuous (see for instance [**DR71**, Theorem2.5]). However, the usual proofs of this fact rely strongly on Haar measure and do not apply to nonlocally compact groups. We see here how to prove this in general, this is based on the following well-known lemma.

Lemma 1.2. *Let $\phi \in \mathcal{CB}(G)$, and consider the map $T_\phi\colon G \to \mathcal{CB}(G)$ defined as $T_\phi(g) = L_g(\phi)$. Then:*

(1) *$\phi \in \mathcal{LUC}(G)$ if and only if T_ϕ is norm-continuous.*
(2) *If $\phi \in \mathcal{WAP}(G)$, then T_ϕ is weakly continuous.*

Proof. Statement (1) follows directly from the definitions.

Now suppose $\phi \in \mathcal{WAP}(G)$ and suppose $(g_i)_{i\in I}$ is a net in G converging to $g \in G$. Let $(g_j)_{j\in J}$ be a subnet of $(g_i)_{i\in I}$. Since ϕ is weakly almost periodic, some subnet of $L_{g_j}\phi$ must converge to some $S\colon \mathcal{CB}(G) \to \mathbb{C}$ in the weak topology. Abusing the notation we keep the name $(L_{g_j}\phi)_{j\in J}$ for this subnet. This in particular means that for any $x \in G$,

$$\lim_{j\in J} \phi(g_j x) = S(x).$$

But since ϕ is continuous and $\lim_{j\in J} g_j = g$ we conclude that $S(x) = \phi(gx)$, and this for all $x \in G$. Hence $S = T_\phi(g)$. Having proved that every subnet of $(T_\phi(g_i))_{i\in I}$ has a subnet weakly convergent to $T_\phi(g)$, we have that $\lim_{i\in I} T_\phi(g_i) = T_\phi(g)$ weakly, and hence that T_ϕ is weakly continuous. $\square$

Lemma 1.3. *If $\phi \in \mathcal{WAP}(G)$, then both $\{R_g f\colon g \in G\}$ and $\{L_g f\colon g \in G\}$ are weakly relatively compact in $CB(G)$.*

Proof. $\square$

Corollary 1.4. *The inclusion $\mathcal{WAP}(G) \subset \mathcal{LUC}(G) \cap \mathcal{RUC}(G)$ holds for every topological group G.*

Proof. That $\mathcal{WAP}(G) \subset \mathcal{LUC}(G)$ follows directly from Lemma 1.2. For the inclusion $\mathcal{WAP}(G) \subset \mathcal{RUC}(G)$, it is necessary to define $S_\phi\colon G \to \mathcal{CB}(G)$ as $S_\phi(g) = R_g(\phi)$. The proof of Lemma 1.2 can then be repeated with S_ϕ, using Lemma 1.3 instead of T_ϕ and the result follows. $\square$

2. Groups of isometries

One common feature of the sets defined in 1.1 is that they all can be realized as algebras of *matrix coefficients* of representations. Before establishing this fact, we recall here the basic terminology concerning Banach representations.

If B is a Banach space, $\mathrm{Is}(B)$ will denote the group of all linear isometries of B, with composition of isometries as group operation. We will always assume that $\mathrm{Is}(B)$ carries the so-called *strong operator topology* (SOT), that is, the topology of pointwise convergence on B. With this operation and topology, $\mathrm{Is}(B)$ becomes a topological group. Another useful topology on the space of operators on B is the *weak operator topology* that is the weak topology generated by the collection of functions $\phi_{\xi,\xi^*}(T) = \xi^*(T\xi)$ when ξ runs over B and ξ^* runs over the conjugate Banach space B^*. If $B = \mathbb{H}$ is a Hilbert space, both (weak operator and strong operator) topologies coincide when restricted to $\mathrm{Is}(\mathbb{H})$. The extension to reflexive spaces of this well-known fact is due to Megrelishvili [**Meg01b**].

A continuous *representation* of a topological group G by isometries of a Banach space B is a continuous group homomorphism $\pi\colon G \to \mathrm{Is}(B)$ of G into the topological group $\mathrm{Is}(B)$ of all linear isometries of B. By a *co-representation* we will refer to a group co-homomorphism (i.e. such that $f(xy) = f(y)f(x)$).

Definition 1.5. Let $\pi\colon G \to \mathrm{Is}(B)$ denote a representation (or a co-representation) of the group G by linear isometries of the Banach space B. If $\xi \in B$ and $\eta^* \in B^*$, the conjugate space of B, the function $\phi_{\pi,\xi,\eta^*}\colon G \to \mathbb{C}$ given by

$$\phi_{\pi,\xi,\eta^*}(g) = \eta^*\,(\,\pi(g)\xi\,)\,,$$

is called the *matrix coefficient* of π given by ξ and η^*.

We will make strong use of two representations (together with its subrepresentations and deformations), the so-called *left-regular and right regular representations* $\mathfrak{L}_G\colon G \to \mathrm{Is}(\ell_\infty(G))$ and $\mathfrak{R}_G\colon G \to \mathrm{Is}(\ell_\infty(G))$ defined on the Banach space $\ell_\infty(G)$ of bounded functions on G with the sup-norm,

$$\mathfrak{L}_G(g)(f) = L_g f \qquad\qquad \mathfrak{R}_G(g)(f) = R_g f$$

Exercise 1. (1) Show that $\mathfrak{L}_G$ and $\mathfrak{R}_G$ are, respectively, a co-representation and a representation of G.

(2) Show that $\mathfrak{L}_G$ defines a continuous co-representation on $\mathrm{Is}(\mathcal{LUC}(G))$ and that all matrix coefficients of this co-representation are left-uniformly continuous (conjugating neighbourhoods, i.e. doing $x^{-1}Ux$ will be necessary at some point). This last assertion is true for all continuous co-representations.
(3) Show that, in general, $\mathfrak{R}_G$ *does not* define a continuous representation of G on $\mathrm{Is}(\mathcal{LUC}(G))$.
(4) Show that $\mathfrak{R}_G$ defines a continuous representation on $\mathrm{Is}(\mathcal{LUC}(G) \cap \mathcal{RUC}(G))$.
(5) Show that a matrix coefficient ϕ_{π,ξ,ξ^*} of a representation $\pi\colon G \to \mathrm{Is}(R)$ on a *reflexive* Banach R space is always weakly almost periodic (use that for fixed ξ^* the map $\xi \mapsto \phi_{\pi,\xi,\xi^*}$ defined on the unit ball of R is continuous). Show as well that $\phi_{\pi,\xi,\xi^*} \in \mathcal{AP}(G)$ if R is *finite dimensional*.
(6) Let $\pi\colon G \to \mathrm{Is}(\mathbb{H}) = \mathcal{U}(\mathbb{H})$ be a representation on a Hilbert space $\mathbb{H}$. Show that every *diagonal matrix coefficient* $\phi_{\pi,\xi,\xi}$ is positive definite (recall that a Hilbert space $\mathbb{H}$ is in duality with itself through the inner product). Show that any coefficient $\phi_{\pi,\xi,\eta}$ is a linear combination of positive-definite functions.

We now show that the sets of functions of subsection 1 are all composed of matrix coefficients (or limits thereof). For $\mathcal{LUC}(G)$ this goes back to Teleman [**Tel57**], in the case of $\mathcal{B}(G)$ and $\mathcal{AP}(G)$ this is the GNS-construction of Gelfand-Naimark-Segal; both cases can be regarded as classical and part of folklore. The case of $\mathcal{WAP}(G)$, due to Megrelishvili [**Meg03**], is more recent and appeals to deep results on factorization of Banach space operators. In the context of compact semitopological semigroups, these results were first applied by Shtern [**Sht94**]. See also [**GMe08**] for a more complete list of function algebras whose elements are representable as matrix coefficients.

Proposition 1.6. *Let G be a topological group and let $f\colon G \to \mathbb{C}$ be a function. Then*

(1) *$f \in \mathcal{RUC}(G)$ if and only if f is a matrix coefficient of some representation $\pi\colon G \to \mathrm{Is}(B)$ of G on a Banach space B.*
(2) *$f \in \mathcal{LUC}(G)$ if and only if f is a matrix coefficient of some co-representation $\pi\colon G \to \mathrm{Is}(B)$ of G on a Banach space B.*
(3) *$f \in \mathcal{WAP}(G)$ if and only if f is a matrix coefficient of some representation $\pi\colon G \to \mathrm{Is}(R)$ of G on a reflexive Banach space R.*

(4) *$f \in \mathcal{WAP}(G)$ if and only if f is a matrix coefficient of some co-representation $\pi\colon G \to \mathrm{Is}(R)$ of G on a reflexive Banach space R.*

(5) *f is positive-definite if and only if $f = \phi_{\pi,\xi,\xi}$ for some representation $\pi\colon G \to \mathrm{Is}(\mathbb{H})$ of G on a Hilbert space $\mathbb{H}$ and some $\xi \in \mathbb{H}$. Also, $f \in B(G)$ if and only if only if $f = \phi_{\pi,\xi,\eta}$ for some representation $\pi\colon G \to \mathrm{Is}(\mathbb{H})$ of G on a Hilbert space $\mathbb{H}$ and some $\xi, \eta \in \mathbb{H}$ (recall that we can identify $\mathbb{H}^*$ with $\mathbb{H}$ itself).*

(6) *$f \in \mathcal{AP}(G)$ if and only if f is the uniform limit of coefficients of representations $\pi_k\colon G \to \mathrm{Is}(\mathbb{H}_k)$ of G on finite-dimensional Hilbert spaces $\mathbb{H}_k$.*

Proof. One direction of all statements is contained in exercise 1, we prove the other direction.

(1) Let $B := \overline{\mathrm{sp}\,\{R_x f\colon x \in G\}}^{\mathcal{RUC}(G)}$, then $\mathfrak{R}_G$ defines a representation on $\mathrm{Is}(B)$. Consider $\delta_{1_G} \in B^*$ defined by $\delta_{1_G}(f) = f(1_G)$. Obviously $\phi_{\mathfrak{R}_G, f, \delta_{1_G}} = f$.

(2) is proved exactly as (1) replacing $\mathcal{RUC}$ by $\mathcal{LUC}$ and $\mathfrak{R}_G$ by $\mathfrak{L}_G$.

(3) This is more delicate. A crucial ingredient is the *factorization theorem* due to Davis, Figiel, Johnson and Pelczynski [**DFJP74**]: *If $K \subset E$ is a weakly compact subset of a Banach space E, then the there is a reflexive Banach space R and a linear one-to-one operator $T\colon R \to E$ such that $K \subset T(B_R)$.*

If $f \in \mathcal{WAP}(G)$, since $f \in \mathcal{RUC}(G)$ as well (Corollary 1.4), we can consider the Banach space B and the representation $\pi\colon G \to \mathrm{Is}(B)$ obtained in (1). Since $\{R_x f\colon x \in G\}$ is a weakly compact subset of B, there is a reflexive Banach space R and a linear one-to-one operator $T\colon R \to B$ such that $\{R_x f\colon x \in G\} \subset T(B_R)$. If, for each $x \in G$, $v_x \in B_R$ denotes the element with $T(v_x) = R_x f$, then a representation $\pi_R\colon G \to \mathrm{Is}(R)$ is defined by

$$\pi_R(g)(v_x) = v_{xg}.$$

Notice that π_R is nothing but the restriction of π to $T(R)$. Clearly $\phi_{\pi_R, v_e, \delta_e \circ T} = f$.

(4) is proved exactly as (3).

(5) This is the classical GNS (Gelfand-Naimark-Segal) construction. If ϕ is a positive-definite function we can define an inner product on $c_{00}(G)$ (the vector space of all finitely supported sequences in $\ell_\infty(G)$) by means of the formula:

$$\langle \xi, \eta \rangle_\phi = \sum_{g,h \in G} \xi(g)\overline{\eta(h)}\phi(gh^{-1}).$$

The quotient $c_{00}(G)/N_\phi$ with $N_\phi = \{\xi \in c_{00}(G)\colon \langle \xi, \xi \rangle_\phi = 0\}$ becomes a pre-Hilbert space, and $\mathfrak{R}_G$ defines a unitary representation on the completion $\mathbb{H}$ of $c_{00}(G)/N_\phi$. The map ϕ is easily seen to be a diagonal matrix coefficient of this representation.

If $\phi \in B(G)$, then ϕ is a linear combination of positive-definite functions $\phi_{\pi_1,\xi_1}, \dots, \phi_{\pi_k,\xi_k}$, and we can recover ϕ as a matrix coefficient of the direct sum $\pi_1 \oplus \cdots \oplus \pi_k$.

(6) is best proven by appealing to semigroup compactifications. We therefore postpone the proof of this fact (see page 22). □

One of the consequences of Proposition 1.6 is to identify the relative sizes of the algebras introduced in subsection 1.

Corollary 1.7. *If G is a topological group,*

$$\mathcal{RUC}(G) \cap \mathcal{LUC}(G) \supset \mathcal{WAP}(G) \supset \mathcal{B}(G) \supset \mathcal{AP}(G).$$

We record here a useful property of weakly almost periodic functions that can be found in almost any reference dealing with these functions, see, for instance, Theorem 4.2.3 of [**BJM89**] for a proof.

Theorem 1.8 (Grothendieck's double limit criterion). *Let $f\colon G \to \mathbb{C}$ be a continuous function. Then $f \in \mathcal{WAP}(G)$ if and only if*

$$\lim_i \lim_j f(g_i h_j) = \lim_j \lim_i f(g_i h_j)$$

whenever (g_i) and (h_j) are sequences such that all limits exist.

Remark 1.9. If $\mathfrak{U}$ and $\mathfrak{V}$ are nonprincipal ultrafilters on $\mathbb{N}$, Grothendieck's double limit criterion can be restated to say: $\phi \in \mathcal{WAP}(G)$ *if and only if*

$$\lim_{n,\mathfrak{U}} \lim_{m,\mathfrak{V}} f(g_n h_m) = \lim_{m,\mathfrak{V}} \lim_{n,\mathfrak{U}} f(g_n h_m)$$

whenever (g_n) and (h_m) are sequences in G.

Examples 1.10. We now give some examples of functions belonging to the above algebra with focus on the case $G = \mathbb{Z}$.

(1) Continuous characters (homomorphisms into the unit circle $\mathbb{T}$) are typical examples of elements of $\mathcal{AP}(G)$ for every topological group G. In the case of $G = \mathbb{Z}$ these are exhausted by the functions $\chi_{t_0}\colon \mathbb{Z} \to \mathbb{T}$, $t_0 \in \mathbb{T}$, given by $\chi_{t_0}(n) = e^{it_0 n}$. Also Fourier-Stieltjes transforms of discrete measures are almost periodic. $\mathcal{AP}(G)$ is actually the closed linear span of all characters when G is a locally compact Abelian group.

(2) A universal source of examples is given by the inclusion $C_0(G) \subset \mathcal{B}(G)$ in case G is locally compact. This is true because all such functions can be approximated by convolutions of compactly supported functions, and these convolutions are in $B(G)$ see [**?**], in particular its Proposition 3.7.
(3) The typical (and universal by Bochner's theorem, see [**Rud90**, Theorem 1.4.3]) example of an element in $B(G) \setminus \mathcal{AP}(G)$ is the Fourier-Stieltjes transform of a continuous measure. For $G = \mathbb{Z}$, an especially simple one is δ_0, the characteristic function of the identity element. $B(G)$ is not uniformly closed and among the nontrivial functions in $\mathcal{B}(G) \setminus B(G)$, we can quote characteristic functions of some interpolation sets, e.g. Sidon sets. Section 3 will be devoted to them, we will see there that characteristic functions of so-called Sidon sets are in $\mathcal{B}(\mathbb{Z})$, while they cannot be in $B(\mathbb{Z})$ unless they are finite see [**DR71**, Section 5.5]).
(4) The question of whether $\mathcal{B}(G)$ and $\mathcal{WAP}(G)$ are distinct was open for some time. The first examples of functions in $\mathcal{WAP}(\mathbb{Z}) \setminus \mathcal{B}(\mathbb{Z})$ were constructed by Rudin [**Rud59**]. The usual examples are characteristic functions of certain interpolation sets. We refer, for the time being to Section 4.3 of [**DR71**] where it is proved that $\mathcal{B}(G) \neq \mathcal{WAP}(G)$ for all locally compact Abelian groups, see [**Cho82**] and [**May97**] for extensions to the noncommutative case.
(5) It is rather easy to find $\mathcal{LUC}$ functions that are not $\mathcal{WAP}$, the characteristic function of $\mathbb{N}$ is an example. It has been proved in [**MPU01**] that for every topological group that is not totally bounded, $\mathcal{WAP}(G) \neq \mathcal{LUC}(G)$.

With the aid of the preceding examples it is easy to deduce that all four algebras are distinct when G is locally compact Abelian and noncompact. If we go beyond this case the picture gets more involved:

Examples 1.11.

(1) For compact groups all four algebras coalesce.
(2) For noncommutative groups the situation is more complicated.

    If G is for instance a noncompact simple Lie group with finite center (such as $SL(2,\mathbb{R})$), then $\mathcal{AP}(G) = \mathbb{C}$ (constant functions) and $\mathcal{WAP}(G) = \mathcal{B}(G) = C_0(G) \oplus \mathbb{C}$ [**Rup84**, Theorem 6.3].

(3) It seems to be unknown whether noncommutative discrete groups always have elements in $\mathcal{B}(G)\setminus C_0(G)$. Chou [**Cho82**] proved nevertheless that $\mathcal{B}(G)\neq \mathcal{WAP}(G)$ in this case. Chou papers [**Cho82**] and [**Cho90**] are a must for a modern point of view on lacunarity on noncommutative groups.

(4) When the group is not locally compact, most of the typical examples mentioned in Example 1.10 are not available. Continuous characters of course are always almost periodic. Important examples of functions in $\mathcal{B}(G)$ are the functions $e^{-\|\cdot\|^\alpha}$ defined on the additive group of $G = L_p(\mu)$ when $1\leq p\leq 2$ and $0<\alpha<1$, [**Sch38**], and Haagerup functions [**Haa79**] on free groups on two generators $F(a,b)$ that have the form $e^{-\lambda|s|}$, $\lambda\in R^+$ (if $s\in F(a,b)$, $|s|$ denotes the length of the reduced word s).

3. The compactifications

We next list the properties of the algebras of section 1 that will lead us to defining the compactifications.

THEOREM 1.12. *Let $\mathcal{X}$ denote any of the sets $\mathcal{LUC}(G)$, $\mathcal{WAP}(G)$, $\mathcal{B}(G)$ or $\mathcal{AP}(G)$, then:*

(1) *$\mathcal{X}$ is a closed vector subspace of $\mathcal{CB}(G)\subset \ell_\infty(G)$, i.e., it consists of bounded functions and is uniformly closed.*
(2) *$\mathcal{X}$ is a conjugation-closed subalgebra of $\mathcal{CB}(G)$.*
(3) *The constant function $\bar{1}$ is in $\mathcal{X}$.*
(4) *$\mathcal{X}$ is translation invariant.*

EXERCISE 2. Prove Theorem 1.12. This should consist of direct applications of the Definitions, Theorem 1.6 and general properties of representations.

The algebras $\mathcal{LUC}(G)$, $\mathcal{WAP}(G)$,$\mathcal{B}(G)$ and $\mathcal{AP}(G)$ are thus that kind of commutative algebras to which Gelfand duality applies: commutative C^*-algebras. If $\mathcal{X}$ is such an algebra the *structure space* or *spectrum* $G^{\mathcal{X}}$ of $\mathcal{X}$ is the space of all continuous linear functionals of $\mathcal{X}$ that are multiplicative, in symbols, $G^{\mathcal{X}}$ is defined as

$$G^{\mathcal{X}} = \{T\colon \mathcal{X}\to\mathbb{C}\colon T\in\mathcal{X}^*, T(f_1f_2)=T(f_1)\cdot T(f_2),\ \text{and } T(1)=1\ \forall f_1,f_2\in\mathcal{X}\}\,.$$

The space $G^{\mathcal{X}}$ will be always equipped with the topology of pointwise convergence on $\mathcal{X}$, i.e., the weak*-topology of $\mathcal{X}^*$.

THEOREM 1.13 (Gelfand duality theorem, see Appendix D of [**Rud90**], for instance). *Let $\mathcal{X}$ be any of the algebras $\mathcal{LUC}(G)$, $\mathcal{WAP}(G)$, $\mathcal{B}(G)$ or $\mathcal{AP}(G)$, then:*

(1) *$G^{\mathcal{X}}$ is a compact Hausdorff space.*
(2) *Evaluations define a continuous semigroup homomorphism $\epsilon_{\mathcal{X}} \colon G \to G^{\mathcal{X}}$ with dense range ($\epsilon_{\mathcal{X}}(g)(f) = f(g)$).*
(3) *Every $f \in \mathcal{X}$ admits a continuous extension $\bar{f} \colon G^{\mathcal{X}} \to \mathbb{C}$ such that $\bar{f} \circ \epsilon_{\mathcal{X}} = f$.*
(4) *There is an isometric isomorphism between the algebras $\mathcal{X}$ and $C(G^{\mathcal{X}}, \mathbb{C})$.*

DEFINITION 1.14. The spectra of the algebras $\mathcal{LUC}(G)$, $\mathcal{WAP}(G)$, $\mathcal{B}(G)$ or $\mathcal{AP}(G)$ will be denoted respectively by $G^{\mathcal{LUC}}$, $G^{\mathcal{WAP}}$, $G^{\mathcal{B}}$ and $G^{\mathcal{AP}}$ and are usually named as $\mathcal{LUC}$-compactification, $\mathcal{WAP}$-compactification, Eberlein-compactification, and Bohr compactification

We have thus constructed four compact spaces related to the four algebras of functions defined in Subsection 1. The chain of inclusions among them presented in Corollary 1.7, clearly induces a chain of quotients:

COROLLARY 1.15. *For any group G there is a family of quotient maps that preserve the embeddings $\epsilon_{\mathcal{X}}$.*

$$G^{\mathcal{LUC}} \to G^{\mathcal{WAP}} \to G^{\mathcal{B}} \to G^{\mathcal{AP}}.$$

We will use the notation $b^{\mathcal{X}}$ for the quotient $b^{\mathcal{X}} \colon G^{\mathcal{X}} \to G^{\mathcal{AP}}$, we will not need a special notation for the rest of the quotients.

These compact spaces are constructed with the hope of finding a convenient playground where good topological properties (compactness) can be used in conjunction with the algebraic, topological or analytic structures present in the underlying group. If $\mathcal{X}$ is any of these algebras we only know so far that there is a continuous map with dense range $\epsilon_{\mathcal{X}} \colon G \to G^{\mathcal{X}}$. In order to exploit the algebraic properties of G in conjunction with compactness we need to introduce some algebraic structure on $G^{\mathcal{X}}$. In the best of the worlds we would like to have that $G^{\mathcal{X}}$ is (1) a compact topological group, that (2) $\epsilon_{\mathcal{X}}$ is a group homomorphism and (3) that it is a homeomorphism (so that we could translate all the structure of G to $\epsilon_{\mathcal{X}}(G)$). We will soon see that this hope is overly optimistic (in fact only totally bounded groups –i.e. dense subgroups of compact groups– can have a compactification with all 3 properties). The rest of these notes will be devoted to understand which portions of (1), (2) and (3) can be obtained.

To begin with we try to move as much of the algebraic structure of G as possible up to $G^{\mathcal{X}}$. First, we must define a binary operation on $G^{\mathcal{X}}$ that extends the operation of G. This will be possible because all four algebras in section 1 are, in the terminology of [**BJM89**], m-left introverted.

LEMMA 1.16. *For any $q \in G^{\mathcal{X}}$ and $\phi \in \mathcal{X}$, consider the maps $T_{q,\phi}, S_{q,\phi}\colon G \to \mathbb{C}$ given by*

$$T_{q,\phi}(g) = q(L_g\phi)), \qquad S_{q,\phi}(g) = q(R_g\phi),$$

then:

(1) *If $\phi \in \mathcal{LUC}(G)$, $T_{q,\phi} \in \mathcal{LUC}(G)$, for all $q \in G^{\mathcal{LUC}}$.*
(2) *If $\phi = \phi_{\pi,\xi^*,\xi}$ for some representation on a* reflexive *Banach space E, then there is $\eta \in E$ such that $T_{q,\phi} = \phi_{\pi,\eta,\xi^*}$.*

PROOF. If $\mathcal{X} = \mathcal{LUC}(G)$, see exercise 3.

To prove (3), take $g_i \in G$, with $q = \lim_i g_i$. The unit ball B_E is $\sigma(E^*, E)$-compact and we can assume that $\lim_i \pi(g_i)\xi = \xi_0 \in B$. Then $T_{q,\phi} = \phi_{\pi,\xi_0,\xi^*}$.

The proof of (2) is very similar. $\square$

COROLLARY 1.17. *Let $\mathcal{X}$ be any of the algebras $\mathcal{LUC}(G)$, $\mathcal{WAP}(G)$, $\mathcal{B}(G)$ or $\mathcal{AP}(G)$. Then $T_{q,\phi} \in \mathcal{X}$ for all $q \in G^{\mathcal{X}}$ and $\phi \in \mathcal{X}$. $S_{q,\phi} \in \mathcal{X}$, if $\mathcal{X} = \mathcal{WAP}(G)$, $\mathcal{B}(G)$ or $\mathcal{AP}(G)$.*

EXERCISE 3.

- Show directly (without resorting to representations) that $T_{q,\phi} \in \mathcal{LUC}(G)$ for any $\phi \in \mathcal{LUC}(G)$.
- If $\phi = \phi_{\pi,\xi^*,\xi}$ for some representation on a Banach space E, then there is $\eta^* \in E^*$ such that $S_{q,\phi} = \phi_{\pi,\eta^*,\xi}$.
- Observe that the previous item does not apply to $S_{q,\phi}$ when $\phi \in \mathcal{LUC}(G)$ but $\phi \notin \mathcal{RUC}(G)$ as such a function cannot be obtained as a matrix coefficient of any representation (albeit it is a matrix coefficient of a *co-representation*) Then, find $\phi \in \mathcal{LUC}(G)$ and $q \in G^{\mathcal{LUC}}$ such that $S_{q,\phi} \notin \mathcal{LUC}(G)$.
- Show in a different way that both $T_{q,\phi}, S_{q,\phi} \in \mathcal{WAP}(G)$, provided $\phi \in \mathcal{WAP}(G)$. Check for instance that:

$$\{R_x T_{q,\phi}\colon x \in G\} \subset \overline{\{R_x\phi\colon x \in G\}}^w \quad \text{and } \{L_x S_{q,\phi}\colon x \in G\} \subset \overline{\{L_x\phi\colon x \in G\}}^w.$$

The same proof would work for $\mathcal{AP}(G)$.

THEOREM 1.18 (Section 2.2 of [**BJM89**]). *Let $(G, \cdot)$ be a group and let $\mathcal{X}$ be one of the algebras in subsection 1. There is then a unique binary operation $*\colon G^{\mathcal{X}} \to G^{\mathcal{X}}$ such that:*

(1) *If* $p, q \in G^{\mathcal{X}}$ *are such that* $p = \lim_i \epsilon_{\mathcal{X}}(g_i)$ *and* $q = \lim_j \epsilon_{\mathcal{X}}(h_j)$*, then:*

$$p * q = \lim_i \left(\lim_j \epsilon_{\mathcal{X}}(g_i \cdot h_j) \right).$$

(2) $(G^{\mathcal{X}}, *)$ *is a semigroup.*

(3) *The continuous map* $\epsilon_{\mathcal{X}} \colon G \to G^{\mathcal{X}}$ *given by Theorem 1.13 is a continuous semigroup homomorphism.*

PROOF. Fix $q \in G^{\mathcal{X}}$. For each $\phi \in \mathcal{X}$, we have by Theorem 1.17 that the function $T_{q,\phi}(g) = q(L_g\phi)$ is in $\mathcal{X}$ as well. Denote by $\overline{T_{q,\phi}}$ its extension to $G^{\mathcal{X}}$.

Define finally $\rho_q \colon G^{\mathcal{X}} \to G^{\mathcal{X}}$ as

$$\rho_q(p)(\phi) = p\left(T_{q,\phi}\right).$$

Then $\phi \circ \rho_q = \overline{T_{q,\phi}}$ for every $\phi \in \mathcal{X}$; since $G^{\mathcal{X}}$ carries the topology of pointwise convergence on $\mathcal{X}$ and $\overline{T_{q,\phi}}$ is continuous, ρ_q is continuous.

The actual definition of $*$ is therefore,

$$p * q = \rho_q(p).$$

By density, this operation must necessarily satisfy the properties of Statement (1). The rest of the Statements are more or less routine, see for instance [**HS98**, Section 4.1]. □

Since the $*$ operation extends the $\cdot$ operation we will no longer use the former notation and use the same symbol for the G and the $G^{\mathcal{X}}$ operation. We will often also drop the $\epsilon_{\mathcal{X}}$ and regard G as a subgroup of $G^{\mathcal{X}}$, but we should beware that G is not a *topological* subgroup of $G^{\mathcal{X}}$, since $\epsilon_{\mathcal{X}}$ is not always a homeomorphism.

EXERCISE 4. Prove directly that $\lim_i \epsilon_{\mathcal{X}}(g_i) = \lim_k \epsilon_{\mathcal{X}}(h_k)$ implies $\lim_i \epsilon_{\mathcal{X}}(g_i g_0) = \lim_k \epsilon_{\mathcal{X}}(h_k g_0)$. With $g_i, h_k, g_0 \in G$.

Try to prove that, conversely, $\lim_i \epsilon_{\mathcal{X}}(g_i) = \lim_k \epsilon_{\mathcal{X}}(h_k)$ implies $\lim_i \epsilon_{\mathcal{X}}(g_0 g_i) = \lim_k \epsilon_{\mathcal{X}}(g_0 h_k)$.

We see here that right-continuity is easier to deal with than left continuity.

In Theorem 1.18 we have actually proved more than we stated:

COROLLARY 1.19. *Let* G *be a topological group and let* $\mathcal{X}$ *be any of the algebras defined in subsection 1.*

(1) *The map* $\rho_q \colon G^{\mathcal{X}} \to G^{\mathcal{X}}$*,* $\rho_q(x) = xq$ *is continuous for every* $q \in G^{\mathcal{X}}$*.* $(G^{\mathcal{X}}, \cdot)$ *is therefore a right topological semigroup.*

(2) *The map* $\lambda_g\colon G^{\mathcal{X}} \to G^{\mathcal{X}}$, $\lambda_g(x) = gx$ *is continuous for every* $g \in G$.

Proof. The first statement has been proved in Theorem 1.18.

For the second, let $g_0 \in G$ be fixed and choose $\phi \in \mathcal{X}$. Since $\mathcal{X}$ is translation invariant, $L_{g_0}\phi \in \mathcal{X}$ and hence extends to $\overline{L_{g_0}\phi}\colon G^{\mathcal{X}} \to \mathbb{C}$. Consider then the map $\lambda_{g_0}\colon G^{\mathcal{X}} \to G^{\mathcal{X}}$ given by

$$\lambda_{g_0}(q)(\phi) = q\left(\overline{L_{g_0}\phi}\right), \qquad \text{for all } q \in G^{\mathcal{X}} \text{ and } \phi \in \mathcal{X}.$$

We deduce that λ_{g_0} is continuous exactly as we did with ρ_q. $\square$

Corollary 1.20. *The quotients of Corollary 1.15*

$$G^{\mathcal{LUC}} \to G^{\mathcal{WAP}} \to G^{\mathcal{B}} \to G^{\mathcal{AP}}$$

are all semigroup homomorphisms.

Example 1.21. *Multiplication on* $\mathbb{Z}^{\mathcal{LUC}}$ *is not commutative.* As a consequence the left multiplication map $\lambda_p\colon \mathbb{Z}^{\mathcal{LUC}} \to \mathbb{Z}^{\mathcal{LUC}}$ $(\lambda_p(q) = p \cdot q)$ cannot be continuous. Therefore $\mathbb{Z}^{\mathcal{LUC}}$ *is not a semitopological semigroup.*

Elements in $\mathbb{Z}^{\mathcal{LUC}}$ may not admit any inverse, so $\mathbb{Z}^{\mathcal{LUC}}$ is not a group either.

Proof. The sets $\mathbb{N}^* = \mathrm{cl}_{\mathbb{Z}^{\mathcal{LUC}}} \mathbb{N} \setminus \mathbb{N}$ and $(-\mathbb{N})^*$ (analogous definition) are disjoint left ideals of $\mathbb{Z}^{\mathcal{LUC}}$: if $x \in \mathbb{N}^*$ and $y \in (-\mathbb{N})^*$, then $xy \in (-\mathbb{N})^*$ while $yx \in \mathbb{N}^*$.

This proves that $\mathbb{Z}^{\mathcal{LUC}}$ is not commutative. Since $\mathbb{Z}$ *is* commutative, λ_x cannot be continuous, for otherwise:

$$xy = \lambda_x(y) \overset{\lambda_x \text{ cont.}}{=} \lim_i \lambda_x(y_i) = \lim_i x + y_i \overset{\mathbb{Z} \text{ commutative}}{=} \lim_i \rho_x(y_i) \overset{\rho_x \text{ cont.}}{=} \rho_x(y) = yx,$$

where the net $(y_i)_i$, $y_i \in G$, has been chosen so that $\lim_i y_i = y$.

$\square$

Exercise 5. Show that $\mathbb{N}^*$ and $(-\mathbb{N})^*$ are disjoint left ideals of $\mathbb{Z}^{\mathcal{LUC}}$.

As we go down in the hierarchy of the algebras, their algebraic structure improves.

Theorem 1.22. *Let G be a topological group.*

(1) $G^{\mathcal{WAP}}$ *and* $G^{\mathcal{B}}$ *are semitopological semigroups (both* λ_p *and* ρ_p *are continuous for all* $p \in G^{\mathcal{WAP}}$ *and* $p \in G^{\mathcal{B}}$*).*
(2) $G^{\mathcal{WAP}}$ *(and hence its quotients* $G^{\mathcal{B}}$ *and* $G^{\mathcal{AP}}$*) is commutative if and only if G is.*
(3) $G^{\mathcal{AP}}$ *is even a topological group.*

(4) *If G is locally compact but not compact, $G^{\mathcal{WAP}}$ is not a topological semigroup, nor a group.*

Proof. To have (1) proved it will suffice to show that $\lambda_q \colon G^{\mathcal{WAP}} \to G^{\mathcal{WAP}}$ is continuous for every $q \in G^{\mathcal{WAP}}$. The proof proceeds exactly as the proof of continuity of ρ_q in Theorem 1.18, replacing $R_{q,\phi}$ by $S_{q,\phi}$. The proof for $G^{\mathcal{B}}$ will be the same.

The proof of Statement (2) is left as an exercise.

To prove (3) we first prove that multiplication is jointly continuous in $G^{\mathcal{AP}}$. Since $G^{\mathcal{AP}}$ has the topology of pointwise convergence on $\mathcal{AP}(G)$ we must show that for any $\phi \in \mathcal{AP}(G)$, the map $(p,q) \mapsto \phi(pq)$ is continuous. Let $(p_i, q_i)_i$ be a net converging to (p,q) in $G^{\mathcal{AP}} \times G^{\mathcal{AP}}$. Since $\phi \in \mathcal{AP}(G)$, we can assume that $R_{q_i}\phi$ is a Cauchy net in the uniform topology. Using that $R_{q_i}\phi$ converges pointwise to $R_q\phi$, we obtain that $R_{q_i}\phi$ converges *uniformly* to $R_q\phi$. Take now i_0 big enough so that

$$\|R_{q_i}\phi - R_q\phi\| \leq \epsilon \qquad \text{for all } i \geq i_0.$$

If we apply this inequality to the elements p_i $(i \geq i_0)$, the result is

$$|\phi(p_i q_i) - \phi(p_i q)| \leq \epsilon, \ \text{ for all } i \geq i_0.$$

And moving i so that $p_i \to p$ and using separate continuity we finally see that

$$|\phi(p_i q_i) - \phi(pq)| \leq 2\epsilon, \ \text{ for all } i \geq i_0.$$

This shows that $\lim_i \phi(p_i q_i) = \phi(pq)$ and, hence, that multiplication on $G^{\mathcal{AP}}$ is jointly continuous. That $G^{\mathcal{AP}}$ is a topological group follows from Exercise 6.

To prove (4) we first assume that G is discrete. As has been said before, the characteristic functions δ_g are in $\mathcal{B}$, this clearly implies that $\epsilon_{\mathcal{B}(G)}(G)$ and $\epsilon_{\mathcal{WAP}}(G)$ are discrete and hence open in $G^{\mathcal{WAP}}$ and $G^{\mathcal{B}}$ (if a compact space has a dense and locally compact topological subspace, the latter must be open) Let now $p \in G^{\mathcal{B}}$ and suppose p has an inverse p^{-1}. We choose a net (g_i) in G with $\lim_i g_i = p^{-1}$. Then, since multiplication is separately continuous $\lim_i (g_i p) = p^{-1}p = 1_G$. Thus $g_i p$ is a net converging to 1_G and, as G is open in $G^{\mathcal{B}}$ we conclude that $g_i p \in G$, for some i. This forces p to be in G and we conclude that the *only elements of $G^{\mathcal{B}}$ or $G^{\mathcal{WAP}}$ that admit an inverse are those of G.*

The proof for locally compact groups is the same once one notices (this is Theorem 2.1) that $\epsilon_{\mathcal{WAP}}(G)$ and $\epsilon_{\mathcal{B}}(G)$ are locally compact when G is locally compact. Exercise 6 finally shows that, not having group structure, $G^{\mathcal{WAP}}$ cannot be a topological semigroup. $\square$

EXERCISE 6. Show that if S is a topological semigroup with identity that contains a dense topological subgroup G, then S itself is a topological group (i.e. every element s has an inverse s^{-1} and the map $s \mapsto s^{-1}$ is continuous

EXERCISE 7. Prove that $G^{\mathcal{WAP}}$ and $G^{\mathcal{B}}$ are commutative if G is Abelian (this is a simple application of Grothendieck's double limit property Theorem 1.8).

Identification of $\mathcal{AP}(G)$ with matrix coefficients of finite-dimensional representations (Statement (4) of Theorem 1.6): Now that we know that $G^{\mathcal{AP}}$ is a compact topological group, we have at our reach all the machinery of the representation theory of compact groups and can finish the proof of Theorem 1.6. We need specifically that every function (in particular the extension $\bar{f}$ of $f \in \mathcal{AP}(G)$ to $G^{\mathcal{AP}}$) on a compact group (such as $G^{\mathcal{AP}}$) can be uniformly approximated by matrix coefficients of unitary representations and that every unitary representation on a compact group is decomposable as a (possibly infinite) direct sum of finite-dimensional representations (see [**Fol95**, Chapter 5]).

4. Some realizations

The compactifications $G^{\mathcal{LUC}}$, $G^{\mathcal{WAP}}$ and $G^{\mathcal{AP}}$ admit several concrete realizations that permit a better understanding of their properties.

We first note that they all are *universal compactifications*. We say that a compactification $(G^{\mathcal{P}}, \epsilon_{\mathcal{P}})$ is universal with respect to a property P if the following two conditions hold:

(1) $G^{\mathcal{P}}$ has property P.
(2) If $\epsilon\colon G \to K$ is a compactification of G (i.e., ϵ is a continuous homomorphism and $\epsilon(G)$ is dense in K) and has property P, then $G^{\mathcal{P}}$ is an extension of K, i.e., there is a map $\epsilon_{\mathcal{P}}^{K}\colon G^{\mathcal{P}} \to K$ making the following diagram commutative:

$$\begin{tikzcd} G \arrow[r, "\epsilon_{\mathcal{P}}"] \arrow[dr, "\sigma"'] & G^{\mathcal{P}} \arrow[d, "\epsilon_{\mathcal{P}}^{K}"] \\ & K \end{tikzcd}$$

Section 3.3 of [**BJM89**] gives conditions for the existence of universal compactifications, the universality of $\mathcal{LUC}(G)$, $\mathcal{WAP}(G)$ and $\mathcal{B}(G)$ is proved in Chapter 4 [loc. cit.].

THEOREM 1.23. *Let G be a topological group:*

(1) *$G^{\mathcal{LUC}}$ is the universal right topological semigroup compactification (is universal with respect to the property of being a right topological semigroup).*
(2) *$G^{\mathcal{WAP}}$ is the universal semitopological semigroup compactification.*
(3) *$G^{\mathcal{AP}}$ is the universal topological (semi)group compactification.*

Observe as well that when G is discrete $G^{\mathcal{LUC}}$ is nothing but βG.

We do not have a description of $G^{\mathcal{B}}$ as a universal compactification. Even if $G^{\mathcal{B}}$ is actually related to the *universal* representation of G, the word universal has a different meaning here.

Denote by $\mathcal{U}(G)$ a set of unitary representations containing exactly one representative of each equivalence class of unitary representations, the universal representation π_u of G is then defined as the direct sum $\pi_u = \oplus_{\pi\in\mathcal{U}(G)}\pi$, it is a unitary representation on the huge Hilbert space $\mathbb{H}_u = \oplus_{\pi\in\mathcal{U}(G)}\mathbb{H}_\pi$, where H_π is the Hilbert space on which π acts.

THEOREM 1.24. *If G is a topological group, $G^{\mathcal{B}}$ can be realized as:*

$$G^{\mathcal{B}} \cong \overline{\pi_u(G)}^{WOT}.$$

EXERCISE 8. Prove Theorem 1.24. It will be necessary to observe that the weak operator topology of $\mathbb{H}_u$ restricted to $\pi_u(G)$ is generated by functions belonging to $B(G)$.

We see in this way $G^{\mathcal{B}}$ as a subsemigroup of the maximal von Neumann algebra $W^*(G)$ of G. The maximal von Neumann algebra of G is defined as $W^*(G) = \overline{\mathrm{sp}(\pi_u(G))}^{WOT}$. When G is locally compact $B(G)$ admits a norm $\|\cdot\|_*$ that makes $(B(G), \|\cdot\|_*)$ into a Banach algebra, $W^*(G)$ can then be realized as the conjugate Banach space of $(B(G), \|\cdot\|_*)$.

One of the best behaved realizations is obtained for $G^{\mathcal{AP}}$ when G is Abelian.

THEOREM 1.25. *If G is an Abelian topological group, $G^{\mathcal{AP}}$ (the Bohr compactification of G) can be realized as the group of* all *characters of the group $\widehat{G}$ of all* continuous *characters of G. In symbols:*

$$G^{\mathcal{AP}} \cong \mathrm{Hom}(\widehat{G}, \mathbb{T}),$$

where $\mathrm{Hom}(\widehat{G}, \mathbb{T})$ *carries the topology of pointwise convergence on $\widehat{G}$.*

PROOF. We need to observe that $\mathrm{Hom}(\widehat{G}, \mathbb{T})$ is a compact Abelian group (it is a closed subgroup of $\mathbb{T}^{\widehat{G}}$) and that, by the Peter-Weyl theorem [**Fol95**, Theorem 5.12], almost periodic functions are uniform limits of linear combinations of continuous characters.

The topological isomorphism, say $\Psi\colon \mathrm{Hom}(\widehat{G}, \mathbb{T}) \to G^{\mathcal{AP}}$, is given by $\Psi(h)(\chi) = \chi(h)$ for every $\chi \in (G^{\mathcal{AP}})\widehat{\ } = \widehat{G}$. We leave as an exercise to check that Ψ is a topological isomorphism. $\square$

EXERCISE 9. Check that the map Φ in Theorem 1.25 is a topological isomorphism. To check that it is onto, observe that elements of $G^{\mathcal{AP}}$ can be regarded as continuous homomorphisms of $(G^{\mathcal{AP}})\widehat{\ }$.

2

Embedding topological groups into their compactifications

In chapter 1 we have introduced several compactifications $(G^{\mathcal{X}}, \epsilon_{\mathcal{X}})$ of topological groups G and found that they admit some traces of the algebraic properties of G. In this Section 3 we study how much of the topological structure of G is transferred by $\epsilon_{\mathcal{X}}$ to $G^{\mathcal{X}}$. The question we will try to answer is for which topological groups G and which algebras $\mathcal{X}$ it is true that the embedding map $\epsilon_{\mathcal{X}}\colon G \to G^{\mathcal{X}}$ is a homeomorphism (onto its image).

1. Basic characterizations

We first present those answers that are more readily obtained.

Theorem 2.1. *Let G be a topological group.*

(1) *$\epsilon_{\mathcal{AP}}$ is a homeomorphism if and only if G is a totally bounded group.*
(2) *$\epsilon_{\mathcal{WAP}}$ and $\epsilon_{\mathcal{B}}$ are homeomorphisms for every locally compact group.*
(3) *$\epsilon_{\mathcal{LUC}}$ is a homeomorphism for every topological group.*

Proof (sketch). To prove (1) we only need to know that (a) a group G is totally bounded if and only if it is (topologically isomorphic to) a subgroup of some compact topological group and (b) that every continuous function on a compact group is almost periodic.

The only ingredient needed to prove (2) is that locally compact groups admit a two sided invariant measure m, their Haar measure, and that for every compact, symmetric neighbourhood U of the identity 1_G, the function $\phi\colon G \to \mathbb{C}$ given by $\phi(g) = m(Ug^{-1} \cap U)$ is in $\mathcal{B}$. This can be checked directly or by using the regular representation $\mathfrak{R}_G$ of G in the unitary group of $L_2(G,m)$. In this latter case we see that ϕ is the matrix coefficient $\phi_{\mathfrak{R}_G,1_U,1_U}$, where 1_U denotes the characteristic function of U. $\square$

Exercise 10. Fill all the gaps left in the preceding proof. In particular, show that every function on a compact group is almost periodic, prove directly that the function ϕ is positive-definite and prove (3).

The embeddability conditions of Theorem 2.1 are better exploited if rephrased in terms of the topology defined by the algebra $\mathcal{X}$ in question. We first leave this concept completely fixed.

Definition 2.2. Let G be a topological group and let $\mathcal{A}$ be a collection of continuous functions $\mathcal{A} \subset C(G, \mathbb{C})$. We say that $\mathcal{A}$ *generates* the topology of G at 1_G if for every neighbourhood U of 1_G there is $\delta_U > 0$ and a function $\phi_U \in \mathcal{A}$ such that

$$\{x \in G\colon |\phi_U(x) - \phi_U(1_G)| < \delta_U\} \subset U.$$

Lemma 2.3. *Let G be a topological group and let $\mathcal{A}$ be a translation-invariant C^*-subalgebra of $CB(G, \mathbb{C})$ that contains the constant function $\overline{1_G}$. The following statements are then equivalent.*

1. *The topology of G is the initial topology defined by the elements of $\mathcal{A}$.*
2. *The elements of $\mathcal{A}$ separate points and closed sets of G.*
3. *The canonical map $\epsilon_{\mathcal{A}}\colon G \to \sigma(\mathcal{A})$ is a homeomorphism.*
4. *The family $\mathcal{A}$ generates the topology of G at 1_G.*

Proof. The equivalence between statements (1), (2) and (3) is very well-known, see for instance [**Eng77**, 2.3.20].

We now prove that (2) implies (4) (this simple fact can be found in several references see for instance [**AMM85**], [**Usp04**, Proposition 2.1] or [**FG**, Lemma 2.3]). Choose for each neighbourhood U of 1_G a function $\phi_U \in \mathcal{A}$ such that $\phi_U(1_G) \notin \mathrm{cl}_G\left(\phi(G \setminus \mathrm{Int}(U))\right)$, where $\mathrm{Int}(U)$ denotes the interior (in the topological sense) of U. Then

$$\inf\{|\phi_U(x) - \phi_U(1_G)|\colon x \in G \setminus \mathrm{Int}(U)\} > 0.$$

It suffices to take then $0 < \delta_U < \{|\phi_U(x) - \phi_U(1_G)|\colon x \in G \setminus \mathrm{Int}(U)\}$.

To see that (4) implies (2), take a closed subset $C \subset G$ and $x \in G$, $x \notin C$. There is then a neighbourhood U of 1_G such that $(xU) \cap C = \emptyset$. Consider $\delta_U > 0$ and ϕ_U as given by Statement (4) and take $\psi = L_{x^{-1}}\phi_U - \overline{\phi_U(1_G)}$. Then $\psi \in \mathcal{A}$ and

$$|\psi(a)| > \delta \quad \text{for all } a \in C, \text{ while } \psi(x) = 0.$$

$\square$

In some special cases, for instance for positive definite functions, statement (4) above can be easily adapted to work with a fixed $\delta < 1$ valid for every U, see [**AMM85**] or [**Gal09**, Lemma 2.1].

1.1. Representability via positive definite and weakly almost periodic functions. Lemma 2.3 has an immediate translation for the algebras we are interested in. After Lemma 2.1 our attention in this regard will be restricted to $\mathcal{B}(G)$ and $\mathcal{WAP}(G)$.

THEOREM 2.4. *Let G be a topological group. The following are equivalent:*

(1) *The map $\epsilon_{\mathcal{B}}\colon G \to G^{\mathcal{B}}$ is a homeomorphic embedding.*
(2) *The topology of G is generated by the functions in $\mathcal{B}$.*
(3) *For every neighbourhood U of 1_G there is a positive-definite function $\phi_U\colon G \to [0, +\infty)$ with $\phi_U(1_G) = 1$, such that*

$$\left\{x \in G\colon 1 - \phi_U(x) < \frac{1}{2}\right\} \subset U.$$

(4) *There is a Hilbert space $\mathbb{H}$ and a topological isomorphism (not necessarily surjective) $j\colon G \to \mathcal{U}(\mathbb{H})$.*

THEOREM 2.5. *Let G be a topological group. The following are equivalent:*

(1) *The map $\epsilon_{\mathcal{WAP}}\colon G \to G^{\mathcal{WAP}}$ is a homeomorphic embedding.*
(2) *The topology of G at 1_G is generated by the family of all weakly almost periodic functions.*
(3) *The topology of G at 1_G is generated by the family of all* positive *weakly almost periodic functions.*
(4) *G is topologically isomorphic to a subsemigroup of a compact semitopological semigroup.*
(5) *There is a reflexive Banach space R and a topological isomorphism $j\colon G \to \mathrm{Is}(R)$.*

EXERCISE 11. Use the results in the preceding chapter to prove theorems 2.4 and 2.5. Some items of the following list of hints may be relevant: (1) positive-definite functions do not constitute a translation-invariant algebra, (2) if $\phi\colon G \to \mathbb{C}$ is positive definite then $|\phi(g)| \leq \phi(1_G) \geq 0$ for all $g \in G$, (3) if ϕ is positive-definite, so is $|\phi(\cdot)|^2$, (4) if $\mathbb{H}$ is a Hilbert space and $\xi \in \mathbb{H}$, the function $T \mapsto e^{\|T\xi - \xi\|}$ is positive definite on $\mathrm{Is}(H)$ (see next exercise) and (5) if a continuous function $\phi\colon G \to \mathbb{C}$ factors through a compact semitopological semigroup, then ϕ is weakly almost periodic, see [**Sht94**] or [**Meg03**] to this respect.

EXERCISE 12. Prove that $(t_1,\dots,t_n)\mapsto e^{-(|t_1|^2+\cdots+|t_n|^2)}$ is a positive definite function on $\mathbb{R}^n$. Show then that $e^{-(|t_1|+\cdots|t_n|)}$ is positive definite as well.

One way to do this is to prove first that $(t_1,\dots,t_n)\mapsto |t_1|^2+\cdots+|t_n|^2$ is negative definite and then use a Theorem of Schoenberg (see e.g. [**BCR84**, Theorem 3.2.2]) to the effect that $e^{-\lambda\psi}$ is positive-definite whenever ψ is positive definite and $\lambda>0$. A function $\psi\colon G\to\mathbb{C}$ is said to be *negative definite* if given $g_1,\dots,g_n\in G$ and complex numbers $\alpha_1,\dots,\alpha_n\in\mathbb{C}$ with $\sum_i\alpha_i=0$, $\sum\limits_{1\le i,j\le n}\alpha_i\overline{\alpha_j}\psi(g_ig_j^{-1})\le 0$.

Another approach for this exercise is by way of Bochner's theorem (see e.g. [**Fol95**, 4.18]): if G is locally compact and abelian, a continuous function $\phi\colon G\to\mathbb{C}$ is positive-definite, if and only if, there is a *positive* measure μ on the character group $\widehat{G}$ such that $\widehat{\mu}=\phi$, where $\widehat{\mu}(t)=\int_{\widehat{G}}\chi(t)d\mu(\chi)$ is the Fourier-Stieltjes transform of μ.

The last statement in each of theorems 2.4 and 2.5 justifies the established terminology that we introduce here.

DEFINITIONS 2.6. A topological group satisfying any (and, hence, all) of the properties of Theorem 2.5 is said to be *reflexively representable.*

A topological group satisfying any (and, hence, all) of the properties of Theorem 2.4 is said to be *unitarily representable.*

EXERCISE 13. Show that unitary representability (resp. reflexive representability) is preserved by arbitrary products

In his seminal paper [**Sch38**], Schoenberg deduced from the facts in exercise 12 that the topological vector spaces $L_p(\mu)$ with $0\le p\le 2$ are unitarily representable. Here by $L_p(\mu)$ we denote the vector space of all μ-measurable functions $f\colon[0,1]\to\mathbb{C}$, where μ is Lebesgue measure. We assume that, for $p>0$, $L_p(\mu)$-spaces come equipped with the metric

$$d_p(f,g)=\left(\int|f(x)-g(x)|^pd\mu(x)\right)^{1/p},$$

while $L_0(\mu)$ is assumed to carry the metric d_0 inducing the topology of convergence in measure:

$$d_0(f,g)=\int\left(1-e^{-|f(x)-g(x)|}\right)d\mu(x).$$

THEOREM 2.7. [**Sch38**] *The topological groups $L_p(\mu)$ are unitarily representable for $0<p\le 2$.*

1.2. Representability via distances. Another, although not that different, way of characterizing representability is through the existence of invariant distances with properties that correspond to positive-definiteness and weak almost periodicity.

The concept of reflexive representability for instance translates into the distance language through stable distances and norms.

DEFINITION 2.8.

- A distance d defined on a set E is called *stable* if for each pair of bounded sequences $\{x_n\}$ and $\{y_m\}$ and of ultrafilters $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathbb{N}$.
$$\lim_{n,\mathfrak{A}} \lim_{m,\mathfrak{B}} d(x_n, y_m) = \lim_{m,\mathfrak{B}} \lim_{n,\mathfrak{A}} d(x_n, y_m).$$
- A Banach space E is called *stable* if for each pair of bounded sequences $\{x_n\}$ and $\{y_m\}$ and of ultrafilters $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathbb{N}$.
$$\lim_{n,\mathfrak{A}} \lim_{m,\mathfrak{B}} \|x_n + y_m\| = \lim_{m,\mathfrak{B}} \lim_{n,\mathfrak{A}} \|x_n + y_m\|.$$

Stable Banach spaces were introduced by Krivine and Maurey [**KM81**] to single out a class of Banach spaces that always contain a copy of ℓ_p for some p:

THEOREM 2.9 (Krivine and Maurey [**KM81**]). *The Banach spaces $L_p(\mu)$ are all stable (and hence reflexively representable by Lemma 2.11). Moreover every stable Banach space contains a copy of ℓ_p for some $1 \leq p < \infty$.*

The connection between stability of Banach spaces and reflexive representability was first noticed by Chaatit [**Cha96**], see also Megrelishvili [**Meg00**]. It has been recently shown that both concepts are essentially equivalent ([**?**] and Theorem 2.11).

Since every nontrivial positive definite mapping satisfies the inequality $\phi(1_G) > 0$ (recall that $|\phi(x)| \leq \phi(1_G) \geq 0$ for every $x \in G$), positive definite functions do not adapt well to define distances. The map $\phi(1_G) - \phi$ does play that rôle, but it is no longer positive-definite. It is the concept of negative-definite function introduced in Example 12 the one that really belongs here. As can be easily checked the function $\phi(1_G) - \phi$ is negative-definite if and only if ϕ is positive-definite.

DEFINITION 2.10. A distance d defined on a set E is said to be *negative-definite* when the kernel $(x, y) \mapsto d(x, y)$ is negative-definite, i.e., when given $g_1, \ldots, g_n \in G$ and complex numbers $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ with $\sum_i \alpha_i = 0$,
$$\sum_{1 \leq i,j \leq n} \alpha_i \overline{\alpha_j} d(g_i, g_j) \leq 0.$$

THEOREM 2.11 ([**?**]). *A metrizable group G is reflexively representable (resp. unitarily representable) if and only if its topology is generated by a left-invariant stable (resp. negative-definite) distance.*

PROOF. One direction is easy, if the topology of G is generated by a left-invariant stable distance d, we only have to observe that the topology of G is generated by the function $E\colon G \to \mathbb{C}$ given by $E(x) = e^{-d(x,1_G)}$ and that this function is weakly almost periodic by Grothendieck's criterion, Theorem 1.8. The same argument works if the topology is generated by a negative definite function using Schoenberg theorem mentioned in Example 12 and characterization 2.4.

The other direction is proved in [**?**, Theorem 3.3]. Assume G is reflexively representable. Since the topology of G is generated by weakly almost periodic functions it is easy to manufacture a function $h(x, y)$ on $G \times G$ as linear combination of weakly almost periodic function that is stable, left-invariant and defining the topology of G. The main difficulty is that this $h(x, y)$ will not satisfy the triangle inequality. The remedy to this `Finish this!!` □

The existence of a stable invariant distance is a rather remarkable feature. Raynaud [**Ray83**] observes that every Banach space admitting such a metric must contain an isomorphic copy of ℓ_p with $1 \leq p < \infty$. Since some reflexive Banach spaces do not contain such subspaces (e.g. Tsirelson's space), this immediately shows that reflexive Banach spaces may fail to be reflexively representable. We will come back to this later.

Another interesting family of unitarily representable groups, that follows from similar principles is made of isometry groups of metric spaces. We quote here the following theorem of Uspenskiĭ.

THEOREM 2.12 (Theorem 3.1 of [**Usp04**]). *Let (M, d) be a metric space. Suppose that there exists a real valued positive-definite function $p\colon \mathbb{R} \to \mathbb{R}$ with $p(0) = 1$, and such that (1) for every $\varepsilon > 0$ we have $\sup\{p(x) : |x| \geq \varepsilon\} < 1$, and (2) for every finite collection $a_1, \dots, a_n \in M$ the symmetric real $n \times n$-matrix $(p(d(a_i, a_j)))$ is positive. Then the topological group $\mathrm{Is}(M)$ of all isometries of M is unitarily representable.*

2. Unitary and reflexive representability as uniform and coarse properties

We will see here that reflexive and unitary representability have a lot to see with the uniform and coarse structure of the group. We begin by specifying the exact meaning of these terms.

Definition 2.13. Let G_1 and G_2 be two topological groups. An injective mapping $f\colon G_1 \to G_2$ is said to be a uniform embedding if both f and f^{-1} are uniformly continuous maps for the respective left-uniformities, i.e., if for every neighbourhood U_2 of 1_{G_2} there is a neighbourhood U_1 of 1_{G_1} such that $xy^{-1} \in U_1$ implies $f(x)f(y)^{-1} \in U_2$ and for every neighbourhood V_1 of 1_{G_1} there is a neighbourhood V_2 of 1_{G_2} such that $f(x)f(y)^{-1} \in V_2$ implies $xy^{-1} \in V_1$.

Definition 2.14. Let (X_1, d_1) and (X_2, d_2) be metric spaces. A map $f\colon X_1 \to X_2$ is said to be a *coarse embedding* if there exist two nondecreasing functions $\varphi_1\colon [0, +\infty) \to [0, +\infty)$ and $\varphi_2\colon [0, +\infty) \to [0, +\infty)$ such that:

(1) $\varphi_1\left(d_1(x, y)\right) \leq d_2\left(f(x), f(y)\right) \leq \varphi_2\left(d_1(x, y)\right)$.
(2) $\lim\limits_{r\to+\infty} \varphi_1(r) = +\infty$.

The problem of finding coarse embeddings of metric spaces in Hilbert or other classes of Banach spaces has been stimulated by the connection, pointed out by Gromov and proven by Yu [**Yu00**], between coarse embeddability of metric spaces in Hilbert space and the Coarse Baum-Connes and Novikov conjectures.

We will see in the present section how the concepts of unitary and reflexive representability seem to provide a natural setting whence uniform and coarse embeddings may both be derived.

We begin with some simple samples of the impact of reflexive and unitary representability in the uniform structure.

Since we pursue to clarify the relation between (unitary or reflexive) representability and the uniform and coarse structure of topological groups, and our definition of coarse embedding has been purely metric, it will be useful to have the metric that generates the topology of $\mathrm{Is}(E)$ clearly identified. This well-known construction is described in the following lemma.

Lemma 2.15. *Let $\{\eta_n\colon n \in \mathbb{N}\}$ denote a dense subset of the unit ball B_E of a separable Banach space E. The strong operator topology of* $\mathrm{Is}(E)$ *is* *then generated by the left-invariant distance:*

$$d_{\mathrm{SOT}}(T,S) = \left(\sum_n \frac{1}{2^n} \|T\eta_n - S\eta_n\|^2 \right)^{\frac{1}{2}}.$$

PROOF. Throughout this proof we will make use of the SOT-neighbourhoods of the identity in $\mathrm{Is}(E)$, $U_{\xi,\varepsilon}$, $\xi \in B$

$$U_{\xi,\varepsilon} = \{T \in \mathrm{Is}(E) \colon \|T\xi - \xi\| < \varepsilon\}.$$

Let $U_{F,\varepsilon} = \cap_{i=1}^n U_{\xi_i,\varepsilon}$ with $F = \{\xi_1, \dots, \xi_n\} \subset E$ denote a basic neighbourhood of the identity in $\mathrm{Is}(E)$. Choose, for each $1 \le i \le n$, $m_i \in \mathbb{N}$ such that $\|\eta_{m_i} - \xi_i\| \le \varepsilon$ and set $n_0 = \max\{m_1, \dots, m_n\}$ A straightforward computation shows that

$$\left\{T \in \mathrm{Is}(E) \colon d_{\mathrm{SOT}}(T,I) \le \frac{\varepsilon}{2^{n_0}}\right\} \subset U_{F,\varepsilon}.$$

For the converse, take any $\varepsilon > 0$ and choose n such that $\sum_{k=n+1}^{\infty} 2/2^k < \varepsilon^2/2$ and consider the neighbourhood of the identity in $\mathrm{Is}(E)$, $U_0 = \bigcap_{k=1}^n U_{\eta_k,\varepsilon^2/2}$. It is clear then that

$$U_0 \subset \{T \in \mathrm{Is}(E) \colon d_{\mathrm{SOT}}(T,I) \le \varepsilon\}.$$

□

For the next Corollary we recall that for a given Banach space B, the Banach space $\ell_p(B)$ is defined as the vector space of all sequences (z_n) with $z_n \in B$ for all n and $\sum_n \|z_n\|^p < \infty$ with the norm $\|(z_n)_n\|_p = \left(\sum_n \|z_n\|^p \right)^{1/p}$.

COROLLARY 2.16. *[***Meg00***] Let E be a separable Banach space. There is a (nonlinear) isometry between* $(\mathrm{Is}(E), d_{\mathrm{SOT}})$ *and the Banach space* $\ell_2(E)$.

PROOF. Use Lemma 2.15, the isometry $\Phi \colon \mathrm{Is}(E) \to \ell_2(B)$ is given by:

$$\Phi(T) = \left(\frac{1}{2^n} T(\eta_n) \right)_n.$$

□

COROLLARY 2.17. *Let E be a separable Banach space. The topological group* $\mathrm{Is}(E)$ *embeds uniformly and coarsely in* $\ell_2(E)$.

PROOF. This is an immediate consequence of Corollary 2.16, for the uniform embedding it is necessary to recall that both involved distances are left-invariant. □

COROLLARY 2.18. *If a separable group G is reflexively representable, then G embeds uniformly and coarsely in a reflexive Banach space.*

PROOF. This follows from Corollaries 2.16 (the coarse part) and 2.17 as soon as one is aware of the duality of $\ell_p(E)$-spaces: $\ell_p(E)^* \cong \ell_{p^*}(E^*)$, with $1/p + 1/p^* = 1$, so that $\ell_2(R)$ is reflexive whenever R is. $\square$

COROLLARY 2.19. *If a separable group G is unitarily representable, then G embeds uniformly and coarsely in ℓ_2.*

The next subsection will be devoted to study the converses to all these corollaries.

2.1. When uniform embedding implies representability. Here we will strongly rely on averaging arguments that will be used to recover the algebraic structure from the uniform one. These averaging processes will be done through invariant means.

DEFINITION 2.20. A topological group is called (left-)*amenable* when $\mathcal{LUC}(G)$ admits a left invariant mean $\mathfrak{m}$, i.e., a continuous linear functional $\mathfrak{m}\colon \mathcal{LUC}(G) \to \mathbb{C}$ with $\mathfrak{m}(1_G) = 1$, $\|\mathfrak{m}\| = 1$ and $\mathfrak{m}(L_g\phi) = \mathfrak{m}(\phi)$ for all $\phi \in \mathcal{LUC}(G)$ and all $g \in G$.

All Abelian topological groups are left-amenable (a consequence of Kakutani fixed point property, see for instance Theorem C1 in the Appendix of [**BL00**]), a typical example of nonamenable discrete group is the free group on two generators. Among not necessarily Abelian amenable groups we can quote solvable groups, locally finite groups and other nonlocally compact groups such as the unitary group of a Hilbert space.

DEFINITION 2.21. Let G_1 and G_2 be topological groups and let $\mathfrak{m}$ be a mean on $\mathcal{LUC}(G_1)$. To every mapping $f\colon G_1 \to G_2$ and every function $\phi\colon G_2 \to \mathbb{C}$ we associate a function $\phi_{f,\mathfrak{m}}\colon G_1 \to \mathbb{C}$, an averaged version of $\phi \circ f$, as follows: first, we define, for each $x \in G_1$, the function $\Phi_{\phi,x,f}\colon G_1 \to \mathbb{C}$, given by $\Phi_{\phi,x,f}(g) = \phi\left(f(xg)f(g)^{-1}\right)$, the function $\phi_{f,\mathfrak{m}}$ is then defined by:

$$\phi_{f,\mathfrak{m}}(x) = \mathfrak{m}(\Phi_{\phi,x,f}).$$

LEMMA 2.22. *Let G_1 and G_2 be topological groups and let $f\colon G_1 \to G_2$ be a uniform embedding. Let in addition $\mathfrak{m}$ denote a mean on $\mathcal{LUC}(G_1)$.*

(1) *If $\phi \in \mathcal{LUC}(G_2)$ then $\phi_{f,\mathfrak{m}} \in \mathcal{LUC}(G_1)$.*

(2) *If the topology of G_2 is generated at 1_{G_2} (see definition 2.2) by a collection of positive functions $\mathcal{A} \subset \mathcal{LUC}(G)$, with $\phi(1_G) = 0$ for all $\phi \in \mathcal{A}$, then the topology of G_1 is generated at 1_{G_1} by the collection $\mathcal{A}_{f,\mathfrak{m}} := \{\phi_{f,\mathfrak{m}} \colon \phi \in \mathcal{A}\}$.*

Proof. We first check that $\phi_{f,\mathfrak{m}}$ is left-uniformly continuous. Take to that end $\varepsilon > 0$. Since ϕ is uniformly continuous there will be a neighbourhood U_2 of the identity in G_2 such that $ab^{-1} \in U_2$ implies $|\phi(a) - \phi(b)| < \varepsilon$. The same argument for f provides a neighbourhood U_1 of the identity in G_1 such that $xy^{-1} \in U_1$ implies $f(x)f(y)^{-1} \in U_2$.

Let now $x, y \in G_1$ be such that $xy^{-1} \in U_1$. Then $f(xg)f(yg)^{-1} \in U_2$ for any $g \in G_1$. This means that $f(xg)f(g)^{-1} \left(f(yg)f(g)^{-1}\right)^{-1} \in U_2$ and hence that $|\Phi_{\phi,x,f}(g) - \Phi_{\phi,y,f}(g)| < \varepsilon$ for every $g \in G$. Thus $\|\Phi_{\phi,x,f} - \Phi_{\phi,y,f}\|_\infty \leq \varepsilon$ and, since $\|\mathfrak{m}\| \leq 1$,

$$\begin{aligned}|\phi_{f,\mathfrak{m}}(x) - \phi_{f,\mathfrak{m}}(y)| &= |\mathfrak{m}\left(\Phi_{\phi,x,f} - \Phi_{\phi.y,f}\right)| \\ &\leq \|\Phi_{\phi,x,f} - \Phi_{\phi,y,f}\|_\infty < \varepsilon.\end{aligned}$$

Statement 2 will be proved in a similar manner. Take some neighbourhood U_1 of 1_{G_1}. Since f is a uniform homeomorphism there is a neighbourhood U_2 of 1_{G_2} such that $x \notin U_1$ implies $f(xg)f(g)^{-1} \notin U_2$ for every $g \in G_1$. Since the topology of G_2 is generated by the functions in $\mathcal{A}$, there are $\phi \in \mathcal{A}$ and $\varepsilon > 0$ such that $\{x \in G_2 \colon \phi(x) < \varepsilon\} \subset U_2$. We deduce therefore that $x \notin U_1$ implies $\phi(f(xg)f(g)^{-1}) \geq \varepsilon$ for every $g \in G$. Thus $\Phi_{\phi,x,f}(g) \geq \varepsilon$ for every $g \in G$ and we deduce that $\phi_{f,\mathfrak{m}}(x) \geq \varepsilon$. Having proved that $\{x \in G_1 \colon \phi_{f,\mathfrak{m}}(x) < \varepsilon\} \subset U_1$ with $\phi \in \mathcal{A}$, the proof is done. $\square$

Lemma 2.23. *Let $G_1 = (G_1, d_1)$ and $G_2 = (G_2, d_2)$ be two metrizable groups equipped with left-invariant metrics d_1 and d_2 and let $f \colon E_1 \to E_2$ be a* uniformly continuous *coarse embedding. Assume in addition that (G_1, d_1) satisfies the following condition:*

($*$) there is a continuous map $j \colon \mathbb{R} \to \mathrm{Aut}(G)$ such that $d_1(j(t)x, 1_{G_1}) \geq t\, d_1(x, 1_{G_1})$

If $\phi \colon G_2 \to \mathbb{C}$ is defined as $\phi(x) = d_2(x, 1_{G_2})$ and $\mathfrak{m}$ is a mean on $\mathcal{LUC}(G_1)$, then the topology of G_1 at 1_{G_1} is generated by the collection of functions $\{\phi_{f,\mathfrak{m}} \circ j(t) \colon t \in \mathbb{R}\}$.

Proof. Take some $\varepsilon > 0$. The definition of coarse embedding provides two nondecreasing functions $\varphi_1 \colon [0, +\infty) \to [0, +\infty)$ and $\varphi_2 \colon [0, +\infty) \to [0, +\infty)$ such that:

(1) $\varphi_1\left(d_1(x, y)\right) \leq d_2\left(f(x), f(y)\right) \leq \varphi_2\left(d_1(x, y)\right)$.

(2) $\lim_{r\to+\infty} \varphi_1(r) = +\infty$.

Suppose now that $x \in G_1$ is such that $d_1(x, 1_{G_1}) \geq \varepsilon$.

To begin with we choose $t_0 \in \mathbb{R}$ such that $\varphi_1(t_0\varepsilon) > 1$ (note that $\lim_{t\to\infty} \varphi_1(t) = +\infty$).

Since φ_1 is nondecreasing, we deduce from property 2.23 that $\varphi_1\big(d_1(j(t_0)x, 1_{G_1})\big) \geq \varphi_1(t_0\varepsilon)$. Our choice of t_0, then implies that $\varphi_1\big(d_1(j(t_0)x, 1_{G_1})\big) \geq 1$. Now d_1 is left-invariant, we thus have that $\varphi_1\Bigg(d_1\big((j(t_0)x)g, g\big)\Bigg) \geq 1$ for every $g \in G_1$. It follows from inequality (1) and the invariance of d_2 that $\phi\Bigg(f\Bigg((j(t_0)x)g\Bigg))f(g)^{-1}\Bigg) \geq 1$ for every $g \in G$. This immediately implies that $\phi_{f,\mathfrak{m}}(j(t_0)x) \geq 1$. Having proved that

$$\Big\{x \in G_1\colon \left(\phi_{f,\mathfrak{m}} \circ j(t_0)\right)(x) < 1\Big\} \subset \Big\{x \in G_1\colon d_1(x, 1_{G_1}) \leq \varepsilon\Big\},$$

and recalling that, by Lemma 2.22, $\phi_{f,\mathfrak{m}}$ is uniformly continuous, the proof is done. $\square$

Corollary 2.24. *Let $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ be quasi-Banach spaces and let $f\colon E_1 \to E_2$ be a uniformly continuous coarse embedding. Let in addition $\mathfrak{m}$ denote a mean on $\mathcal{LUC}(E_1)$. Then, defining $\phi\colon E_2 \to \mathbb{C}$ as $\phi(\xi) = \|\xi\|_2$, the topology of E_2 is generated at 1_{G_2} by the collection of functions $\{\phi_{f,\mathfrak{m}} \circ j(t)\colon t \in \mathbb{R}\}$, where $j(t)$ is defined by $j(t)\xi = t\xi$ for every $t \in \mathbb{R}$ and $\xi \in E_1$.*

Proof. Apply Lemma 2.23 with $j\colon \mathbb{R} \to \mathrm{Aut}(E_1)$ defined by $j(t)\xi = t\xi$ for every $\xi \in E_1$. $\square$

We now apply Lemmas 2.22 and 2.23 to see how unitary and reflexive representability are related to the coarse and uniform structure of amenable groups. Our first result comes essentially from [**AMM85**].

Theorem 2.25. *Let G_1 and G_2 be topological groups with G_1 left-amenable and let $f\colon G_1 \to G_2$ be a uniform embedding. If G_2 is unitarily representable, then G_1 is unitarily representable.*

Proof. First, choose an *invariant* mean $\mathfrak{m}$ on $\mathcal{LUC}(G_1)$.

By Theorem 2.4 the collection of positive functions

$$\mathcal{A} = \{1 - \phi\colon \phi \text{ is a real-valued positive definite function on } G_2 \text{ with } \phi(1_{G_2}) = 1\}$$

defines the topology of G_2 at 1_{G_2}.

By Statement 2 of Lemma 2.22, applied to the collection $\mathcal{A}$, the functions $\{\psi_{f,\mathfrak{m}}\colon \psi \in \mathcal{A}\}$ define the topology of G_1 at 1_{G_1}. If $\psi = 1 - \phi$, then $\psi_{f,\mathfrak{m}} = 1 - \phi_{f,\mathfrak{m}}$. Again by Theorem 2.4, it will suffice to see that $\phi_{f,\mathfrak{m}}$ is positive-definite for every positive-definite function $\phi\colon G_2 \to \mathbb{C}$.

Suppose $\phi\colon G_2 \to \mathbb{C}$ is positive-definite and let $x_1, \ldots, x_n \in G$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ be given.

Observe that, by invariance of $\mathfrak{m}$,

$$\sum_{i,j} \alpha_i \overline{\alpha_j} \phi_{f,\mathfrak{m}}(x_i x_j^{-1}) = \sum_{i,j} \alpha_i \overline{\alpha_j} \mathfrak{m} \left(L_{x_j} \Phi_{x_i x_j^{-1}, f} \right). \tag{2.1}$$

Since $L_{x_j} \Phi_{x_i x_j^{-1}, f}(g) = \phi\left(f(x_i g) f(x_j g)^{-1}\right)$ we turn our attention to the family $f(x_1 g), \ldots, f(x_n g)$. As ϕ is positive-definite we obtain that, for any $g \in G$,

$$\sum_{i,j} \alpha_i \overline{\alpha_j} \phi\left(f(x_i g) f(x_j g)^{-1}\right) \geq 0,$$

whence $\mathfrak{m}\left(\sum_{i,j} \alpha_i \overline{\alpha_j} L_{x_j} \Phi_{x_i x_j^{-1}, f}\right) \geq 0$. Thus, continuing in (2.1),

$$\sum_{i,j} \alpha_i \overline{\alpha_j} \phi_{f,\mathfrak{m}}(x_i x_j^{-1}) = \mathfrak{m}\left(\sum_{i,j} \alpha_i \overline{\alpha_j} L_{x_j} \Phi_{x_i x_j^{-1}, f} \right) \geq 0,$$

and $\phi_{f,\mathfrak{m}}$ is positive-definite. □

COROLLARY 2.26. *Let G_1 and G_2 be uniformly homeomorphic topological groups. G_1 is unitarily representable if and only G_2 is unitarily representable.*

The following lemma in conjunction with Theorem 2.25 yields a uniform classification of unitary representability.

COROLLARY 2.27. *Let G be an amenable topological group. Then G is unitarily representable if and only if G embeds uniformly in ℓ_2^κ for some cardinal κ.*

PROOF. If G is unitarily representable, there is a topological isomorphism of $j\colon G \to \mathcal{U}(\mathbb{H})$. The Hilbert space $\mathbb{H}$ is isomorphic to a Hilbert sum, $\oplus_\kappa \ell_2$. Since $\mathcal{U}(\oplus_\kappa \ell_2)$ is topologically isomorphic to a subgroup of $\prod_\kappa \mathcal{U}(\ell_2)$, we conclude with Lemma 2.17 that G embeds uniformly in ℓ_2^κ.

If conversely G embeds in ℓ_2^κ we only have to apply Theorem 2.25. □

Concerning reflexive representability we can now improve Theorem 2.11. This will also result in an extension of Raynaud's results [**Ray83**] where it is proved that a Banach space that admits a uniform embedding into a superstable Banach space (see [**Ray83**] for the definition) always admits an invariant uniformly equivalent stable distance.

Corollary 2.28. *An amenable topological group G_1 admits an uniform embedding into a metrizable group (G_2, d_2), with d_2 being a stable and left-invariant metric, if and only if G_1 admits an equivalent left-invariant distance that is stable.*

In virtue of Lemma 2.23, coarse properties can also be added to the above consequences of unitary and reflexive representability. The coincidence between uniform embeddability and coarse embeddability in Hilbert space has been noted by Randrianarivony [**Ran06**]. We see here how an isomorphic property as unitary representability can work as the *right* bridge between both properties. We need the following lemma.

Lemma 2.29. *Let (X, d) be a metric space. If (X, d) admits a coarse embedding into a Hilbert space $\mathbb{H}$, then (X, d) admits a uniformly continuous coarse embedding into $\mathbb{H}$.*

Proof. The coarse embedding turns into a Lipschitz embedding (`TRUE???`) when restricted to a 1-net in G. Since a Lipschitz map with values in Hilbert spaces can always be extended to a Lipschitz map on a metric superspace, it is easy to see that this restriction that is Lipschitz equivalent to a 1-net in □

Corollary 2.30. *Let $G = (G, d)$ be a left-amen able metrizable group with property $(*)$ of Lemma 2.23. The following statements are equivalent:*

1. *G is unitarily representable.*
2. *G embeds uniformly in a Hilbert space.*
3. *G embeds coarsely in a Hilbert space*

Proof. Properties (1) and (2) are equivalent by Corollary 2.27. Theorem 2.17 shows that (1) implies (3) as well. Assume now (3). By Lemma 2.29 we can assume that E admits a uniformly continuous coarse embedding into a Hilbert space. Lemma 2.23 proves next that the topology of G is defined by the collection of functions

$$\{\psi_{f,\mathfrak{m}} \circ j(t) \colon \ \psi = 1 - \phi, \text{ positive definite function on } G_2, \text{ with } \phi \geq 0 \text{ and } \phi(1_{G_1}) = 1,\ t \in \mathbb{R}\}$$

where $j(t)$ is, for each $t \in \mathbb{R}$, an automorphism of G. By Theorem 2.22 and the proof of) Lemma 2.25 the functions $\psi_{f,\mathfrak{m}}$ are all continuous and positive definite and so will be their composition with $j(t)$. It follows from Theorem 2.4 that G is unitarily representable. □

3. Groups that are unitarily or reflexively representable: dual groups of Banach spaces

We have thus far identified a number a topological groups that are unitarily or reflexively representable such as locally compact groups or additive groups of $L_p(\mu)$-spaces. We now try to give a broader picture of these classes.

Some classes of groups that have an "approximately locally compact" behaviour are also unitarily representable. The following theorem is proved in [**Ban01**].

THEOREM 2.31 (Banasczcyk [**Ban01**]). *Nuclear groups are unitarily representable*

Theorem 2.32 is trivial for nuclear locally convex spaces, for they are projective limits of Hilbert spaces. This proof would extend to general nuclear groups through the structure theorem of [**Gal00**] if quotients of Hilbert spaces by weakly closed subgroups were unitarily representable. Since we do not know whether this is true or not, we have to rely on the more technical proof of [**Ban01**].

QUESTION 1. Let E be a Hilbert space and let H be a weakly closed subgroup of E, is E/H unitarily representable?

We now develop a mechanism to obtain other classes of Abelian unitarily representable groups. It is based on results of Fonf, Johnson, Plichko and Shevchyk concerning the factorization of compact operators into ℓ_p-sums of finite dimensional Banach spaces. Some definitions are first required:

DEFINITIONS 2.32.

- A Banach is called and $\mathcal{L}_p$ space if the metric structure of its finite dimensional spaces is "close" to that of ℓ_p^n (spaces of dimension n with ℓ_p-norm), see [**BL00**, Appendix F]. Of course, $L_p(\mu)$ spaces are $\mathcal{L}_p$ for every μ and $C(K)$-spaces are $\mathcal{L}_\infty$.
- Johnson's space $\mathcal{R}$ is defined as the ℓ_2-sum

$$\mathcal{R} = \left(\sum_n M_n \right)_2 ,$$

where $(M_n)_n$ is a family of finite-dimensional Banach spaces, dense for the Banach-Mazur distance. Roughly speaking this means that the family (M_n) contains approximately a copy of every finite-dimensional spaces up to isometry, see [**Joh71**] for the actual definition.

THEOREM 2.33 (Theorem 2.1 and Remark 4.5 of [**FJPS06**]). *Let E denote a Banach space and let K be a compact subset of E.*

(1) *If E has the approximation property*[1]*, there is a 1-1 compact operator $T_K\colon \mathcal{R} \to E$ with $K \subset T(B_R)$.*
(2) *If E is an $\mathcal{L}_\infty$-space, there is a collection (C_n) of finite dimensional spaces with each C_n isometric to $\ell_\infty^{(k(n))}$ and a 1-1 compact operator $T_K\colon \left(\sum_n C_n\right)_0 \to E$ with $K \subset T(B_E)$.*

PROOF. The proof of this theorem can be found in [**FJPS06**, Theorem 2.1]. We provide a proof of Statement (2) for the particular case in which E admits a monotone basis $(e_n)_n$ such that the spaces $E_n = \mathrm{sp}(e_1, \ldots, e_n)$ are isometrically isomorphic to ℓ_∞^n. Recall that a sequence $(e_n)_n$ is said to be a *monotone basis* when every $x \in E$ can be uniquely represented as $x = \sum_{n=1}^\infty \alpha_n x_n$ with $\alpha_n \in \mathbb{C}$ in such a way that the partial sum projections $P_n\colon E \to \mathrm{sp}(e_1, \ldots, e_n)$ all have norm $\|P\| \leq 1$. $C(K)$ spaces with K compact and metric all have a monotone basis with $E_n = \mathrm{sp}(e_1, \ldots, e_n)$ isometrically isomorphic to ℓ_∞^n.

Since compact subsets of Banach spaces are contained in closed convex hulls of null-sequences, we can assume for simplicity that K itself is a null-sequence $K = \{z_n \colon n < \omega\}$.

Define $E_n := \mathrm{sp}(e_1, \ldots, e_n)$. The partial sum projection will be denoted by $P_n\colon E \to E_n$. For each n, put $z_n = \sum_k \alpha_{n,k} e_k$. The sum being convergent, it will be possible to find for every n and M, an index $k_{M,n}$ such that

$$\left\| \sum_{k=r}^{\infty} \alpha_{n,k} e_k \right\| \leq \frac{1}{2^{2M}} \qquad \text{for all } r \geq k_{M,n}.$$

Since, on the other hand, $z_n \to 0$, there is n_M with $\|z_n\| \leq 2^{-2M}$ for every $n \geq n_M$. Choosing $k_j = \max\{k_{j,n} \colon n < n_j\}$ (we set $k_0 = 1$), we obtain that

$$\left\| \sum_{k=k_j+1}^{k_{j+1}} \alpha_{n,k} e_k \right\| \leq \frac{1}{2^{2j}} \qquad \text{for every } n \geq 1 \text{ and } j \geq 1. \tag{3.1}$$

[1] A Banach space E is said to have the *approximation property* if there is a sequence of operators $T_n\colon E \to E$ with finite dimensional range such that $\lim_n \|x - T_n x\| = 0$ for every $x \in E$. All classical Banach spaces have the approximation property.

We now define $M_0 = E_{k_1}$, $M_n = E_{k_{n+1}}$ for $n \geq 1$ and $F = \left(\sum_{j=0}^{\infty} M_j\right)_0$.

The operator $T_K \colon F \to E$ is then defined as

$$T_K\left((y_j)_j\right) = \sum_{j=0}^{\infty} \frac{1}{2^j} y_j.$$

Since the norms $\|y_j\|_E (= \|y_j\|_{M_j})$ are bounded, it is clear that the above sum is absolutely convergent in E.

To see that T_K is compact we take $\varepsilon > 0$ and check that $T_K(F_1)$ can be covered by finitely many translates of the ball of radius 2ε centered at the origin.

Choose j_0 such that $\sum_{j>j_0} \frac{1}{2^j} \leq \varepsilon$.

Since M_{j_0} is finite dimensional, it is possible to find $a_1, \ldots, a_N \in M_{j_0}$ such that

$$v \in M_{j_0}, \quad \|v\| \leq 2 \implies \text{there is } 1 \leq k \leq N \text{ with } \|v - a_k\| \leq \varepsilon.$$

Now, let $v := (v_j)_j \in F_1$, i.e., such that $\|v_j\|_E \leq 1$ for every j. Since $\sum_{j \leq j_0} \frac{1}{2^j} v_j \in M_{j_0}$ and $\|\sum_{j\leq j_0} \frac{1}{2^j} v_j\| \leq \sum_{j \leq j_0} \frac{1}{2^j} \leq 2$, there must be some k with $\|\sum_{j \leq j_0} \frac{1}{2^j} v_j - a_k\| \leq \varepsilon$. We obtain finally that

$$\|T_K(v) - a_k\| = \left\| \sum_j \frac{1}{2^j} v_j - a_k \right\| \leq$$
$$\left\| \sum_{j \leq j_0} \frac{1}{2^j} v_j - a_k \right\| + \left\| \sum_{j > j_0} \frac{1}{2^j} v_j \right\| \leq 2\varepsilon.$$

To see that $K \subset T_K(F_1)$, we consider an arbitrary $z_n \in K$. Choosing

$$w_j = 2^j \sum_{k=k_j+1}^{k_{j+1}} \alpha_{n,k} e_k, \qquad \text{for } j \geq 0,$$

we have by (3.1) that

$$\|w_j\|_{M_j} \leq \frac{1}{2^j}, \qquad \text{for every } j \geq 1.$$

This means both that $(w_j)_j \in \left(\sum_j M_j\right)_0 = F$ and that $\|(w_j)_j\|_\infty = \sup_{j\geq 0} \|w_j\|_{M_j} \leq 1$. Since $z_n = T_K\left((w_j)_j\right)$ we have that $K \subset T_K(F_1)$ and the proof is done. $\square$

REMARK 2.34. The paper [**FJPS06**] contains a much stronger conclusion than Statement 2 in Theorem 2.34. Actually for any given $\mathcal{L}_\infty$ space Y there is a compact operator $T_{\mathrm{K}}\colon Y \to E$ with $K \subset T(B_Y)$.

Theorem 2.34 has a strong impact in the unitary and reflexive representability of duals of Banach spaces with the compact-open topology. Since this topology is the natural one in the duality theory of topological groups we use the term dual group for these sapces.

DEFINITION 2.35. If B is a Banach space, we will use the symbol $\widehat{B}$ and the term *dual group of* B to refer to the additive group of the dual space B^* (the vector space of all continuous linear functionals) equipped with the topology of convergence on compact subsets of B.

REMARK 2.36. It is a well known fact that the dual group of B is topologically isomorphic to the character group of the additive group of B, see for instance [**Ban91**, Proposition 2.3].

THEOREM 2.37 (Theorem 2 in [**FG**] and Theorem 3 of [**Gal09**]). *Let B denote a Banach space and let $\widehat{B}$ denote its dual group.*

(1) *If B has the approximation property, then $\widehat{B}$ is reflexively representable.*
(2) *If B is an $\mathcal{L}_\infty$-space, then $\widehat{B}$ is unitarily representable.*

PROOF. Let Let $\mathcal{K}(B)$ denote a set that is cofinal in the family of all compact subsets of B (ordered by inclusion). For the rest of the proof R will denote an ℓ_p-sum of finite dimensional spaces $(\sum_n M_n)_p$. For the proof of (1), R will indeed refer to Johnson's $\mathcal{R}$ space. For the proof of (2), p will equal 0 and each space M_n will be isometric to $\ell_\infty^{k(n)}$ for some $k(n)$ (were we using the results of [**FJPS06**] in all their force, see the remark after Theorem 2.34, we could have taken $R = c_0$).

For each $K \in \mathcal{K}(B)$ we consider now the compact operator provided by Theorem 2.34, $T_{\mathrm{K}}\colon R_{\mathrm{K}} \to B$, that is defined on a copy R_{K} of R.

Define then

$$\Phi\colon \widehat{B} \to \prod_{K\in\mathcal{K}(B)} R^*_{\mathrm{K}}$$

as the product $\Phi = \prod_{K\in\mathcal{K}(E)} T^*_K$, where again R^*_{K} represents a copy of R^* and $T_{\mathrm{K}}*\colon \widehat{B} \to R^*_{\mathrm{K}}$ is the operator adjoint to T_{K}.

This map is easily seen to be one-to-one.

To see that Φ is continuous we begin by observing that

$$T^*_{\mathrm{K}}\left(\left\{\xi^* \in B^*\colon \xi^*\left(T_{\mathrm{K}}(B_{R_{\mathrm{K}}})\right) \subset \mathbb{D}_\varepsilon\right\}\right) \subset \varepsilon B_{R^*_{\mathrm{K}}}, \tag{3.2}$$

where $\mathbb{D}_\varepsilon$ denotes the disc of the complex plane of radius ε. Taking into account that T_{K} is a compact operator, (3.2) leads us to deduce that $T_{\mathrm{K}}*\colon \widehat{B} \to R_{\mathrm{K}}^*$ is continuous.

To see that Φ is an open mapping onto its image, choose a compact subset $K_0 \subset B$ and let $U = \{\xi^* \in B^*\colon \xi^*(K_0) \subset \mathbb{D}_\varepsilon\}$ denote the corresponding basic neighbourhood of 0 in $\widehat{B}$. The covering property of T_{K} then implies that $T_{K_0}(B_{R_{K_0}}) \subset K_0$ and hence that

$$\left\{ \left(\xi_{\mathrm{K}}^*\right)_{K\in\mathcal{K}(B)} \in \prod_{K\in\mathcal{K}(B)} R_{\mathrm{K}}^*\colon \xi_{K_0}^* \in B_{R_{K_0}^*} \right\} \cap \Phi(B^*) \subset \Phi(U).$$

It follows that $\Phi(U)$ is a neighbourhood of 0 in $\Phi(B^*)$ and, since U was arbitrary, that Φ is relatively open.

We have thus that Φ is a linear homeomorphism of $\widehat{B}$ onto a subspace of $\prod_{K\in\mathcal{K}(E)} R_K^*$.

Proof of (1): If $R = \mathcal{R}$, each R_K^* is linearly isometric to $(\sum_n M_n^*)_{\ell_2}$. Since each M_n^* is finite dimensional, it is stable. The ℓ_2-sum of stable Banach spaces is known to be stable (see [**KM81**, Théorème II.1]). It follows from Theorem 2.11 that $\prod_{K\in\mathcal{K}(E)} R_K^*$ and, as a consequence, that $\widehat{B}$ is reflexively representable.

Proof of (2): If $R_{\mathrm{K}} = (\sum_n M_n)_0$ with each space M_n isometric to $\ell_\infty^{k(n)}$ for some $k(n)$, we have that $R_{\mathrm{K}}^* = (\sum_n M_n^*)_1$. As M_n^* is isometric to $\ell_1^{k(n)}$, we conclude that $R_{\mathrm{K}}^* \subset \ell_1$. Therefore $\widehat{B}$ embeds in a product of ℓ_1's and hence is unitarily representable. □

We have already seen that nuclear groups are unitarily representable. An immediate consequence of Theorem 2.38 is that all Schwartz locally convex spaces are reflexively representable. Schwartz locally convex spaces constitute a class of spaces where some of the classical theorems of Analysis takes place and that still retains some of the flavour of finite dimensional spaces. See [**Jar81**] or [**Hor66**] for the basic properties of Schwartz locally convex spaces. See [**ACDT07**] as well for the introduction of the concept in the topological group setting.

The essential example of Schwartz space is the dual group of a Banach space. It is essential because every Schwartz locally convex space is linearly homeomorphic to a subspace of a power of $\widehat{\ell_1}$, the dual group of ℓ_1 (this was obtained independently by Jarchow [**Jar73**] and Randtke [**Ran73**]). Since ℓ_1 has the approximation property, Statement (1) of Theorem 2.38 yields the following:

Corollary 2.38. *The additive group of a Schwartz locally convex vector space is reflexively representable.*

Another consequence, or rather the same, of Theorem 2.38 concerns free topological groups. Theorem 2.38 can be directly applied to free locally convex spaces and free Abelian topological groups on compact spaces. If X is a completely regular space, the free locally convex space $L(X)$ and the free Abelian topological group $A(X)$ on X are obtained by providing the free vector space on X (resp. the free Abelian group on X) with a locally convex vector space topology (resp. a topological group topology) such that every continuous function $f\colon X \to E$ into a locally convex space (resp. a topological grouop) can be extended to a continuous linear map $\bar{f}\colon L(X) \to E$ (resp. to a continuous homomorphism): free topological groups are used to get *linear structure* were it is lacking. The space of continuous functions $C(X, \mathbb{R})$ and the group of continuous functions $C(X, \mathbb{T})$ can then be identified (via the restriction mapping) with the dual vector space and the character group of $L(X)$ and $A(X)$, respectively. The topology of $L(X)$ happens then to be that of uniform convergence on equicontinuous pointwise bounded subsets of $C(X, \mathbb{R})$ and the topology of $A(X)$ the topology of uniform convergence on equicontinuous subsets of $C(X, \mathbb{T})$, see [**Usp83**], and [**Pes95**] for proofs of these facts.

The elements of $L(X)$ can also be seen as linear combinations of point-mass measures on X. Since the topology of $L(X)$ is the topology of uniform convergence on equicontinuous pointwise bounded (=relatively compact) sets, $L(X)$ is a subspace of the dual group of $C(X, \mathbb{R})$. The latter is usually identified with the space $M_c(X)$ of measures with compact support on X, in a natural way. By Tkachenko-Uspenskiĭ's theorem [**Tka83, Usp90**], the topology of $A(X)$ is actually the topology inherited from $L(X)$ and, *a fortiori* from $M_c(X)$.

When X is compact, $C(X, \mathbb{R})$ is a Banach space and $M_c(X)$ (the subindex is usually dropped in this case) is its dual group, the following is then a simple Corollary of Theorem 2.39.

Corollary 2.39 (Corollary 6 of [**Gal09**]). *The additive group of $M_c(K)$ and its subgroups $L(K)$ and $A(K)$ are unitarily representable for every compact Hausdorff space K.*

3.1. Free topological groups. Theorem 2.40 shows that free Abelian topological groups on compact spaces are unitarily representable. Restricting to the compact case is unnecessary as Uspenskij [**Usp08**] has shown.

What makes the unitary representability of free topological groups important is their universal property: every topological group is a quotient of a free topological group. For instance the solution to an important problem in descriptive set theory(see [**GP03**] for its formulation) would follow immediately if the following question had a positive answer:

QUESTION 2 ([**GP03**] and [**Pes07b**]). Is every Polish topological group a quotient of a subgroup of the unitary group of some Hilbert space?

In view of the preceding remarks the answer to this question would be in the affirmative if we had an answer to the following one:

QUESTION 3 (Pestov, see for instance [**Pes07b**]). Are free topological groups unitarily representable?

There are however paths to Question 2 other than the answer to Question 3. Indeed Gao and Pestov were able to obtain the commutative version of question 2 without appealing to free Abelian topological groups (the answer came nevertheless from the same circle of ideas, as it made strong use of free Banach spaces).

EXERCISE 14. Show that every *Abelian* topological group is a quotient of a unitarily representable group. The path followed in [**GP03**] is (1) embed the Abelian topological G group in a product of metrizable groups $\prod_i M_i$, (2) show that each metrizable group M_i is the quotient of some Banach space (use here the Free Banach or Lipschitz-free space on M_i), (3) every separable Banach space is a quotient of ℓ_1 and (4) ℓ_1 is unitarily representable.

We now see that all free Abelian topological groups are unitarily representable, this is a Theorem of Uspenskiĭ.

THEOREM 2.40 (Theorem in [**Usp08**]). *Let X be a completely regular topological space. The free locally convex space $L(X)$ over X is then unitarily representable*

PROOF (SKETCH). A fundamental tool for the proof is the Bartle-Graves's theorem (see, for instance, [**BL00**, Proposition 1.19]): *every surjective linear operator $T\colon E \to F$ between Banach spaces has a continuous left-inverse, i.e., there is $S\colon F \to E$ such that $T \circ S = id_F$.*

Let now $f\colon X \to B$ be a continuous map with values in a Banach space B. Applying the Bartle-Graves theorem to the quotient $p_f\colon \ell_1 \to B$ we get a continuous map $s_f\colon B \to \ell_1$ with $p_f \circ s_f = id_B$. Let finally $E_f\colon L(X) \to \ell_1$ be the linear extension of $s_f \circ f$ to $L(X)$.

If $\mathcal{F}$ denotes the family of all continuous functions on X with values on some Banach space, then $(\prod_f E_f)(L(X))$ is a subspace of a power of ℓ_1 and is linearly homeomorphic to $L(X)$. $\square$

EXERCISE 15. Complete the preceding proof. First justify why we only take care of mappings with range in a Banach space. Then prove with rigor that $(\prod_f E_f)(L(X))$ is topologically isomorphic to $L(X)$, note to that end that every continuous function of $f\colon X \to B$, extends to a continuous linear map defined on $(\prod_f E_f)(L(X))$, viz. to the restriction composition of the f-projection with (the restriction of) p_f.`check!!`

We remark that questions 2 and 3 remain open for nonAbelian topological groups. We quote here the following Conjecture of Uspenskiĭ:

CONJECTURE 1 (Conjecture 4.1 of [**Usp08**]). *The free topological group $F(X)$ is topologically isomorphic to a subgroup of* $\mathrm{Is}(M)$*, the isometry group of a metric subspace of $L_1(\mu)$.*

4. When $\mathcal{WAP}(G)$ and $\mathcal{B}(G)$ are small: nonrepresentable groups

Not all groups are reflexively representable. The extreme case is represented by the group $\mathrm{Homeo}_+[0,1]$ of all orientation preserving homeomorphisms of the interval $[0,1]$. The proof of this fact can be found in Megrelishvili's paper [**Meg01a**] or in its generalization in [**GMe08**] to the effect that this group does not admit nontrivial representations in classes of Banach spaces larger than reflexive ones.

THEOREM 2.41 (Megrelishvili [**Meg01a**]). *Every weakly almost periodic function on* $\mathrm{Homeo}_+[0,1]$ *is constant.*

The group $\mathrm{Homeo}_+[0,1]$ is a large group that is far from Abelian. In general the algebras $\mathcal{WAP}(G)$ and $\mathcal{B}(G)$ tend to be richer when G is Abelian and it can be expected that all Abelian groups are unitarily or reflexively representable (whether this is true was asked by several authors, like Ruppert [**Rup84**], Shtern [**Sht94**], Megrelishvili [**Meg01a, ?**] or Pestov [**Pes07b**]). We will see here that the answer to both questions is negative and that extreme examples are available also in the Abelian case, at least for $\mathcal{B}(G)$

We begin by finding some classical Banach spaces that are not unitarily representable. This depends on turning the topological (or rather, uniform) information provided by positive definite functions into linear information.

To proceed in this way we need the basic structure of commutative von Neumann algebras (see [**Zim90**] or [**KR97**, Section 9.4]). The essential fact of this theory is that every commutative von Neumann algebra of operators $\mathcal{A}$ is isometrically isomorphic to $L_\infty(X,\mu)$ for some measure space (X,μ) and that this isomorphism can also be chosen to preserve the action of the algebras as algebras of operators. More concretely, if $\mathcal{A}$ is an algebra of operators on the Hilbert space $\mathbb{H}$, there are isometric isomorphisms $\mathfrak{V}\colon \mathcal{A} \to L_\infty(X,\mu)$ and $\mathfrak{V}_{\mathbb{H}}\colon L_2(X,\mu) \to \mathbb{H}$ such that $\mathfrak{V}(T)f = T(\mathfrak{V}_{\mathbb{H}}(f))$. It is important to recall here that $L_\infty(X,\mu)$ acts on the Hilbert space $L_2(X,\mu)$ by multiplication operators: every $f \in L_\infty(X,\mu)$ induces a multiplication operator $M_f\colon L_2(X,\mu) \to L_2(X,\mu)$, $M_f(g) = fg$. The unitary group of $L_\infty(X,\mu)$ is then $L_\infty(X,\mu,\mathbb{T})$, the multiplicative group of all $f \in L_\infty(X,\mu)$ with with $|f| = 1$, a.e. We will write $L_0(X,\mu,\mathbb{T})$ to denote the group $L_\infty(X,\mu,\mathbb{T})$ equipped with the weak operator topology that on this set coincides with the topology of convergence in measure.

Lemma 2.42. *Let E be a topological vector space. Every positive definite $\phi\colon E \to \mathbb{C}$ induces continuous homomorphisms $V_\phi\colon E \to L_0(X,\mu,\mathbb{T})$ (multiplicative) and $U_\phi\colon E \to L_0(X,\mu)$ (additive) such that:*

$$\phi(g) = \int_X V_\phi(g)(x)\,d\mu(x), \qquad \textit{and} \qquad e^{iU_\phi} = V_\phi.$$

Proof (sketch). By Theorem 1.6 there is a unitary representation $\pi\colon E \to \mathcal{U}(\mathbb{H})$ such that $\phi = \phi_{\pi,\xi,\xi}$ for some $\xi \in \mathbb{H}$. Since E is Abelian, the smallest weak operator closed subalgebra $\mathcal{A}$ of $B(\mathbb{H})$ (the algebra of all bounded operators on $\mathbb{H}$) that contains $\pi(E)$ is commutative `add a reference!`. By the preceding remarks there is a isometric isomorphism $\mathfrak{V}\colon \mathcal{A} \to L_\infty(X,\mu)$ and $f \in L_2(X,\mu)$ so that $\phi = \phi_{\mathfrak{V}\circ\pi,f,f}$.

The claimed map V_ϕ will be the composition $\mathfrak{V}\circ\pi$. Since $\pi(E)$ is made of unitaries, so will be the range of V_ϕ, hence $V_\phi(E) \subset L_0(X,\mu,\mathbb{T})$.

The representation π in this disguise will be our V_ϕ.

It is a consequence of the resolution theorems for one-parameter groups of operators that every continuous homomorphism such as $V_\phi(E) \subset L_\infty(X,\mu,\mathbb{T})$ has a "logarithm" that produces U_ϕ, see [**Ban91**, Section 4]. □

A direct proof of the existence of U_ϕ in the above lemma can be found in [**AMM85**] or [**BL00**, Proposition 8.7]

Corollary 2.43. *If a metrizable topological vector space E is unitarily representable, then it is topologically isomorphic to an additive subgroup of $L_0(X,\mu)$, for some compact X and some Borel measure μ on X and to a multiplicative subgroup of $L_0(X,\mu,\mathbb{T})$.*

THEOREM 2.44. *An infinite-dimensional $L_p(\mu)$-space is unitarily representable if and only if* $0 \leq p \leq 2$.

IDEA OF PROOF. We have already noted that $L_p(\mu)$ is unitarily representable for $0 \leq p \leq 2$, see te remarks after exercise 12.

For $p > 2$, the infinite dimensional spaces $L_p(\mu)$ have *cotype* p (see section III.A of [**Woj91**] for the notion and results about cotype) while Banach spaces linearly homeomorphic to subspaces of $L_0(X, \mu)$ all have cotype 2, [**BL00**, Proposition 8.17]. □

We will next pursue this line of thought. To get more subtle examples about reflexive representability or unitary representability we will need to resort to deeper results. These examples all involve the Banach space c_0.

5. c_0 as source of examples

The Banach space c_0 was the first Banach space that was found to be nonuniformly embeddable in ℓ_2, this was achieved by Enflo in [**Enf69**]. In this section we will obtain some stronger results that will lead to a different proof of Enflo's theorem, Theorem 2.49. All these results provide new clues suggesting that $\mathcal{WAP}(c_0)$ is quite poor and hence a good source of (negative) examples.

While dealing with the Banach space c_0, the sequence with one in the nth place and zero otherwise will be denoted by e_n.

We first need a couple of results reminiscent of the notion of cotype. The first of them can be found for instance as Theorem 13 of [**Dil01**] or as Proposition 8.16 of [**AMM85**]

THEOREM 2.45 (Nikishin factorization theorem). *Let X be a Banach space. Every continuous linear operator $T\colon X \to L_0(\mu)$ factorizes through L_q for each $0 < q < 1$, i.e. there are continuous linear operators $S_1\colon X \to L_q$ and $S_2\colon L_q \to L_0$ such that $S_2 S_1 = T$.*

THEOREM 2.46 (Theorem 4.3 of [**KMS93**]). *For every bounded operator $T\colon c_0 \to L_q$, $0 < q \leq 2$, $\sum \|T(e_n)\|^2 < \infty$.*

With this ingredients we get the following version of Corollary 3 of [**Gal09**].

COROLLARY 2.47. *If $\phi \in B(c_0)$ is positive definite with $\phi(0) = 1$, then* $\lim_n \phi(e_n) = 1$.

PROOF (SKETCH). Let $\phi\colon c_0 \to \mathbb{C}$ be positive definite and continuous. By Lemma 2.43, there is a continuous linear operator $U_\phi\colon c_0 \to L_0(\mu)$. Let $c_0 \overset{S_1}{\to} L_q \overset{S_2}{\to} L_0$ be a factorization of the operator U_ϕ as in Theorem 2.46. By Theorem 2.47, $\sum_n \|S_1(e_n)\|_q^2 = M < \infty$, hence $\lim_n U_\phi(e_n) = \lim_n S_2(S_1(e_n)) = 0$. From $U_\phi(e_n) \to 0$ it is easy to deduce that $\lim_n \phi(e_n) = 0$. □

From Corollary 2.48 we deduce immediately that positive definite functions do not generate the topology of c_0, thus:

COROLLARY 2.48 (Enflo [**Enf69**]). *c_0 is not unitarily representable.*

But much more can be deduce from Corollary 2.48, for instance that some Abelian groups have trivial $\mathcal{B}(G)$:

THEOREM 2.49 (Banaszczyk [**Ban83**]). *There is a closed subgroup H of c_0 with $\mathcal{B}(c_0/H)$ consisting solely of constant functions.*

PROOF (SKETCH). Enumerate a countable dense subgroup of c_0 as $(v_n)_n$, and do it in such a way that $v_n \in \mathrm{sp}(e_1, \ldots, e_{n-1})$, and define $H = \mathrm{sp}(e_n + v_n)_n$. Since every element of $h \in H$ is of the form $\sum_j z_j(e_j + v_j)$ with $z_j \in \mathbb{Z}$, we have that $\|h_1 - h_2\| \geq 1$ for every $h_1, h_2 \in H$, $h_1 \neq h_2$ and hence H is discrete, and closed.

Suppose now that π is a unitary representation of c_0/H on a Hilbert space $\mathbb{H}$. This gives rise to a representation $\tilde{\pi}$ of c_0 such that $\tilde{\pi}(H) = Id$. From Theorem 2.48 we deduce that $\tilde{\pi}(e_n)$ must converge to the identity operator Id in the weak operator topology, and also in the strong operator topology, for both topologies coincide in the unitary group.

Since $\tilde{\pi}(H) = Id$,

$$\tilde{\pi}(v_n) = \pi(e_n)^{-1}. \tag{5.1}$$

Taking into account that $\tilde{\pi}(e_n) \to Id$ we deduce that $\lim_n \tilde{\pi}(v_n) = Id$. But (v_n) was a dense subgroup of c_0, what means that $\tilde{\pi}(v_n)$ is dense in $\tilde{\pi}(c_0)$. We conclude that $\tilde{\pi}(c_0) = \pi(c_0/H) = \{Id\}$ and, since π was arbitrary, that c_0/H does not have nonconstants unitary representations. □

Summarizing: we now know that Abelian groups can behave as badly as possible concerning positive definite functions, they may even have none at all. On the other hand, we also have found certain families of groups where positive-definite functions work well: locally compact groups, dual groups of $\mathcal{L}_\infty$ spaces, nuclear groups. All these are examples of Schwartz groups. We also know that all Schwartz spaces are reflexively representable. We cannot therefore leave this picture without wondering whether all Schwartz spaces are unitarily representable. Notice also that we have not as yet any example of Abelian groups that are not reflexively representable let alone any one satisfying Theorem 2.50 for weakly almost periodic functions.

5.1. Unitarily representable topologies on c_0. We consider now c_0 as a vector subspace of $\widehat{\ell_1}$. The topology that c_0 inherits (i.e. that of uniform convergence on compact subsets of ℓ_1) is the finest one among all possible Schwartz locally convex vector space topologies that are coarser than the norm topology, [**?**, Corollary 14.5]. We denote this locally convex vector space as $\mathcal{S}(c_0)$. It is known, [**Ran73**] or [**Jar73**], that every other locally convex Schwartz space is a subspace of a power of $\mathcal{S}(c_0)$.

Neighbourhoods of 0 in $\mathcal{S}(c_0)$ are determined by sequences of numbers going to 0: given such a sequence $\overline{\alpha} = (\alpha_n)_n$, consider the neighbourhood

$$U_{\overline{\alpha}} = \left\{ (x_n)_n \in c_0 \colon |x_n| \le \frac{1}{|\alpha_n|}, \text{ for every } n \right\}. \tag{5.2}$$

The sets $U_{\overline{\alpha}}$ (with $\alpha \in c_0$) then constitute a neighbourhood basis at 0 of $\mathcal{S}(c_0)$, see [**Ran73**, Corollary 3].

Denote now by c_{00} the space of all sequences with finitely many nonzero terms.

Lemma 2.50. *If $\phi\colon c_{00} \to \mathbb{C}$ is norm-continuous and positive-definite then it is also continuous in the $\sigma(c_{00}, \ell_2)$-topology.*

Proof (sketch). We proceed as in Theorem 2.48, associating to every such ϕ a linear map $T\colon c_{00} \to L_0$ and splitting T through L_p as $S_1 S_2$. Since (Theorem 2.47) $(S_1(e_n)) \in \ell_2$, it is then easy to check that for every neighbourhood U of 1 in $\mathbb{C}$, the set $\phi^{-1}(U)$ contains a neighbourhhod determined by $S_1(e_n)$ (roughly speaking, if $\sum |a_n S_1(e_n)|$ is small, $T((a_n)) = S_2 \sum a_n S_1(e_n)$ and $T((a_n)$ is again small. □

This will be a valuable tool to answer one of our remaining questions.

Theorem 2.51. *The Schwartz space $\mathcal{S}(c_0)$ is not unitarily representable.*

Proof. Consider the subspace c_{00} of c_0. Since every positive-definite mapping on c_{00} is continuous in the $\sigma(c_{00}, \ell_2)$ topology we only have to observe that the $\sigma(c_{00}, \ell_2)$ topology is strictly weaker than the topology of $\mathcal{S}(c_0)$. But this follows easily from the description of neighbourhoods given in (5.2), simply take $\bar{\alpha} \in c_0$, $\bar{\alpha} \notin \ell_2$, then $U_{\bar{\alpha}}$ does not contain any $\sigma(c_{00}, \ell_2)$-neighbourhood. □

We finally remark that some Abelian groups are not reflexively representable. This is the result of adapting the following theorem of Raynaud, [**Ray83**].

THEOREM 2.52 (Theorem 2.4 of [**FG**]). *The additive group of c_0 is not reflexively representable.*

PROOF. Suppose d is a stable, invariant distance generating the topology of c_0.

Let $s_n = (1/n)\,(e_1 + \cdots + e_n)$ denote the summing basis of c_0 normalized dividing by n.

We will find a sequence $(h_k) \subset c_0$ with $\lim_k \phi(h_k) = \phi(0)$ for every $\phi \in \mathcal{WAP}(G)$ and $\|h_k\| = 1$ for every k. This will show that the topology of c_0 is not generated by weakly almost periodic functions.

Let $\phi \in \mathcal{WAP}(G)$ and let an integer k be fixed. Choose nontrivial ultrafilters $\mathfrak{U}_1, \ldots, \mathfrak{U}_{2k}$, then

(5.3)

$$L_{\phi,k} := \lim_{n_1,\mathfrak{U}_1} \lim_{n_2,\mathfrak{U}_2} \cdots \lim_{n_{2k-1},\mathfrak{U}_{2k-1}} \lim_{n_{2k},\mathfrak{U}_{2k}} \phi\left(\sum_{j=1}^{k} s_{2j} - \sum_{j=1}^{k} s_{2j-1}\right)$$

(5.4)

$$\overset{\phi\in\mathcal{WAP}}{=} \lim_{n_1,\mathfrak{U}_1} \lim_{n_3,\mathfrak{U}_3} \cdots \lim_{n_{2k-1},\mathfrak{U}_{2k-1}} \lim_{n_2,\mathfrak{U}_2} \lim_{n_4,\mathfrak{U}_4} \cdots \lim_{n_{2k-2},\mathfrak{U}_{2k-2}} \lim_{n_{2k},\mathfrak{U}_{2k}} \phi\left(\sum_{j=1}^{k} s_{2j} - \sum_{j=1}^{k} s_{2j-1}\right).$$

Note that in the second limit we have shuffled the order of summation. One can see that the limit is preserved after this shuffling by applying a variant of Grothendieck's double limit, Theorem 1.8, see [**BL00**, Lemma 9.19] for the validity of this variant. Observe that for $n_1 < n_2 < \ldots n_k < \ldots < n_{2k}$,

$$\| \sum_{j=1}^{k} s_{n_{2j}} - \sum_{j=1}^{k} s_{n_{2j-1}} \| = \frac{1}{k}. \tag{5.5}$$

As a consequence of (5.3), $\lim_k L_{\phi,k} = \phi(0)$ for any ϕ.

Applying the second limit, (5.4), we can find for each k a collection of indices $n_1 < n_3 < \cdots < n_{2k-1} < n_2 < \cdots < n_{2k}$ big enough for the following inequality to hold:

$$\left| \phi\left(\sum_{j=1}^{k} s_{n_{2j}} - \sum_{j=1}^{k} s_{n_{2j-1}}\right) - L_{\phi,k} \right| < \frac{1}{k}. \tag{5.6}$$

Using these indices we define now $h_k = \sum_{j=1}^k s_{n_{2j}} - \sum_{j=1}^k s_{n_{2j-1}}$. Since $\lim_k L_{\phi,k} = \phi(0)$. It follows from (5.6) that $\lim_k \phi(h_k) = \phi(0)$. Observe on the other hand that (recall that, for every k, $n_1 < n_3 < \cdots < n_{2k-1} < n_2 < \cdots < n_{2k}$)

$$\|h_k\| = \|\sum_{j=1}^{k} s_{n_{2j}} - \sum_{j=k+1}^{2k} s_{n_{2j-1}}\| = 1. \tag{5.7}$$

□

6. Concluding remarks

We have found that additive groups of locally convex Schwartz spaces occupy the border between unitarily and reflexively representable groups, they are always reflexively representable but may fail to be unitarily representable. In general, topological groups may have very few representations on Hilbert or even reflexive Banach spaces. The additive group of c_0 for instance is not reflexively representable and one of its quotients does not admit any nontrivial unitary representation. Thus, commutativity alone does not seem to improve much this sort of representation-theoretic behaviour. It should be remarked nevertheless that, to this author knowledge, no example of an Abelian topological group admitting no nontrivial representation on reflexive Banach spaces is known. That is why we ask whether there is such a counterpart for Theorem 2.50:

Question 4. Is there an Abelian topological group G with no nonconstant weakly almost periodic functions?

Recall that examples of nonAbelian topological groups with no nonconstant weakly almost periodic functions do exist (Theorem megrhomeo).

The relation of reflexive and unitary representability with the uniform or coarse classification of topological groups, gives some other results and opens several other questions.

By Corollary 2.30 and 2.31, an amenable metrizable topological group is unitarily representable if and only if it embeds coarsely or uniformly in a Hilbert space and it is reflexively representable if and only if it embeds uniformly in stable metric group. Two open ends are clearly left open here.

Question 5. Suppose (G, d) embeds coarsely in a stable metric space (X, ρ), is G then reflexively representable? What if (G, d) and (X, ρ) are Banach spaces?

For noncommutative groups, more precisely for nonamenable groups, even the simplest relations between uniform embeddability and representability are not known:

QUESTION 6 (Megrelishvili). If G is unitarily representable and H is uniformly homeomorphic to G (for the left uniformities), is H necessarily unitarily representable?

Since reflexively representable groups embed uniformly in reflexive Banach spaces, we may wonder about the relation existing between reflexive representability and embeddings in reflexive Banach spaces. The difficulty resides in finding a specific class of functions that makes a topological group uniformly embeddable in a reflexive Banach space.

Raynaud [**Ray83**] proves that a Banach space E admitting a stable distance uniformly equivalent to the norm distance, then E must contain some *linear* copy of ℓ_p for some $1 \leq p < \infty$. Since some reflexive Banach spaces do not contain any linear copy of ℓ_p (as for instance Tsirelson space, see [**BL00**, Section 10.5]), we deduce from 2.31 that

THEOREM 2.53 (Theorem 4.2 of [**?**]). *There are reflexive Banach spaces that are not reflexively representable.*

Although the class of topological groups that embed uniformly in a reflexive Banach space is strictly larger than that of reflexively representable groups, it does not contain all of Abelian topological groups. This is deep result of Kalton.

THEOREM 2.54 (Kalton, [**Kal07**]). *The Banach space* c_0 *does not embed uniformly in any reflexive Banach space.*

Observe that theorem 2.53 now follows directly from Theorem 2.55 if one applies Theorem 2.18.

The last section of [**Kal07**] contains several questions related to the structure of Banach spaces that embed coarsely (or uniformly) in stable metric spaces. They are obviously linked to the structure of reflexively representable Banach spaces through Theorem 2.31. Concerning coarse embeddings some data is obviously still missing.

QUESTION 7. Let (X, d) and (Y, ρ) be metric spaces. If (X, d) embeds coarsely in (Y, ρ), is there always a coarse embedding of (X, d) into (Y, ρ) that is *uniformly continuous*? What if (X, d) is a Banach space?

3

Size and complexity of compactifications. Interpolation sets

The compactifications we have introduced in section 3 satisfy some of our demands at the cost of being quite complicated objects; we examine in this Section some proofs of this complexity; this will be done through the introduction of a class of sets that are key instruments in the analysis of semigroup compactifications.

This sets have as main property that their closure in the compactification is homeomorphic to $\beta\mathbb{N}$. We will thus be taking $\beta\,\mathbb{N}$ as a measure of what we mean a complicated space. This is admittedly vague and not much rigorous but we only want to get an idea of the complexity of our spaces and do not pretend to dwell on establishing precise hierarchy.

We already know that for G discrete, $G^{\mathcal{LUC}}$ can be realized as the very well-known object βG. Even if $G^{\mathcal{WAP}}$ and $G^{\mathcal{B}}$ should in principle be simpler objects, they are however not so well understood. We will find proofs of this fact very often in the remainder of these notes.

We remark that if Section 2 has been devoted to relate the topology of the group G and the topology induced by the compactifications, the analysis of this Section can be seen as relating the topological group *uniformity* of G with the uniformity induced by the compactifications.

Let us now introduce the sets that will serve us to illustrate how big and complex these structures are:

Definition 3.1. Let G be a group and let $\mathcal{X}$ denote a subalgebra of $\ell^\infty(G)$, we say that a subset A of G is an $\mathcal{X}$**-interpolation set** when every bounded function $f\colon A \to \mathbb{C}$ admits a continuous extension $\bar{f} \in \mathcal{X}$.

Examples 3.2 (Some special interpolation sets). Let G be a topological group. Interpolation sets for some of the algebras of Section 1 have been heavily studied and have been baptized:

(1) If G is discrete, then every set is an $\mathcal{LUC}(G)$-interpolation set. For nondiscrete G, every *uniformly discrete* subset of G will be an $\mathcal{LUC}(G)$-interpolation sets.

(2) There is no exact reference in the literature on $\mathcal{WAP}(G)$-interpolation sets. Close relatives are Ruppert's *finite translation sets* [**Rup85**] also called R_W*-sets* by Chou [**Cho90**]. These are WAP-interpolation sets E whose characteristic function $\chi_E \in WAP(G)$ (we will call them uniformly approximable $\mathcal{WAP}(G)$-interpolation sets, see Theorem 3.6).

(3) The $\mathcal{B}(G)$-interpolation sets are called Sidon sets. Picardello [**Pic73**] uses the name *weak*-Sidon sets. He reserves the name Sidon set to the interpolation sets that appear replacing $B(G)$ by another smaller algebra, for amenable G (in particular, for Abelian G) both clases coincide.

(4) The $\mathcal{AP}(G)$-interpolation sets are called I_0-sets, this name goes back to the work of Hartman and Ryll-Nardzewski [**HRN64**]. It is interesting to observe that every bounded function f on an I_0-set admits an almost periodic extension $\bar{f}$ with absolutely convergent Fourier series[**Kah66**].

The presence of $\mathcal{X}$-interpolation sets is a measure of how complex the compactification but also of how rich the algebra $\mathcal{X}$ is, recall that for to be A is an $\mathcal{X}$-set lots of very different functions must come from elements of $\mathcal{X}$. It is also a notion of independence (if S is a linearly independent subset of a vector space E, every function on S extends to a *linear* map on E). From the topological point of view this is reflected in the presence of copies of $\beta\mathbb{N}$.

LEMMA 3.3. *Let G be a topological group. For a set $A \subset G$ the following assertions are equivalent:*

(1) *A is an $\mathcal{X}$-interpolation set.*

(2) *Every $f\colon A \to \mathbb{C}$ admits a continuous extension $\bar{f}$ to the compact space $G^{\mathcal{X}}$.*

$$\begin{tikzcd} G \arrow[r, "\epsilon_{\mathcal{X}}"] & G^{\mathcal{X}} \arrow[d, "\bar{f}"'] \\ A \arrow[u, "incl"] \arrow[r, "f"] & \mathbb{C} \end{tikzcd}$$

(3) *The closure $\mathrm{cl}_{G^{\mathcal{X}}} A$ of A in $G^{\mathcal{X}}$ is homeomorphic to βA.*

(4) *If $A_i \subseteq A$, $i = 1, 2$ are disjoints subsets of A, then $\mathrm{cl}_{\epsilon_{\mathcal{X}}(\mathcal{X})} A_1 \cap \mathrm{cl}_{\epsilon_{\mathcal{X}}(\mathcal{X})} A_2 = \emptyset$.*

PROOF.

$2 \Longleftrightarrow 1$ By Theorem 1.13, the evaluation map $\phi \mapsto T_\phi$ is an isometric isomorphism isomorphism from $\mathcal{X}$ onto $\mathcal{X} = C(G^{\mathcal{X}}, \mathbb{C})$ ($T_\phi(p) = \phi(p)$ for all $p \in G^{\mathcal{X}}$).

If A is an $\mathcal{X}$-interpolation set, every $f\colon A \to \mathbb{C}$ is the restriction to A of some $\phi_f \in \mathcal{X}$. The evaluation T_{ϕ_f} is a continuous function on $G^{\mathcal{X}}$. We may define in that case $\bar{f} = T_{\phi_f}$.

The same equality defines an element $\phi_f \in C(G^{\mathcal{X}}, \mathbb{C}) = \mathcal{X}$ associated to f when Condition 2 holds. Both ϕ_f and $\bar{f}$ extend f to an element of $\mathcal{X}$ and to a continuous function on $G^{\mathcal{X}}$ respectively.

$2 \Longleftrightarrow 3 \Longleftrightarrow 4$ These conditions are equivalent for every subset of a compact space, see [**Dug66**] .

□

1. How interpolation sets look like: lacunarity

Lacunary sequences are at the root of the concept of interpolation set. A sequence a_n of integers is called a *lacunary sequence* if

$$\frac{a_{n+1}}{a_n} \geq \lambda > 1 \qquad\qquad \text{for some } \lambda \text{ and every } n.$$

Lacunary sequences were introduced by Hadamard to discuss a particular sort of homogeneity in the behaviour of certain trigonometric series, see [**Kat76**, Chapter 5]. In [**Str63**] Strzlecki proved that lacunary sequences are I_0 sets, see [**KR99**] for a recent proof.

We will prove here the technically simpler fact that lacunary sequences with $\lambda > 4$ are I_0-sets. In fact the bigger λ is the greater degree of "independence" the set has [**GH06a**]. From lemma 3.3 it clearly follows that a sequence $(a_n)_n$ is an I_0-set if given two closed disjoint intervals I_1 and I_2 in $\mathbb{T}$ and an arbitrary arrangement $(I_n)_n$ of the intervals I_1 and I_2 (so that I_n is either I_1 or I_2) there is always an almost periodic function χ with $\chi(n) \in I_n$.

Example 3.4 (Lacunary sequences). *Lacunary sequences are always I_0-sets*

Proof for the case $|z_{n+1}/z_n| \geq 4$. Let I_1 and I_2 be two symmetric closed intervals in $\mathbb{T}$ of *arc length* $1/3$ centered in 1 and -1 respectively. It is crucial to observe that both intervals will contain at least one n-rooth of the unity for every $n \geq 2$. Let $\{I_n\}$ be an arbitrary arrangement of I_1 and I_2.

We initially choose an interval $\mathbf{J_1}$ of arc length $1/3|z_1|$ with $z_1(\mathbf{J}_1) = \{t^{a_1} : t \in \mathbf{J}_1\} = I_1$.

The set $a_2(J_1)$ will be an arc of length $\frac{|z_2|}{3|z_1|}$, and will cover the whole $\mathbb{T}$ once and at least one sixth of I_1 or I_2 twice. For some $\mathbf{t_2} \in J_1$, $z_2(t) \in I_2$, if the interval centered in t_2 of arc length $1/3|z_2|$ is contained in J_1 we choose this interval to be $\mathbf{J_2}$, otherwise we still can move to the other corner and have room to define an interval $\mathbf{J_2} \subset \mathbf{J_1}$ of the the same length such that z_2 maps $\mathbf{J_2}$ into I_2. Following in this manner we get a sequence $\mathbf{J_n}$ of nested intervals in $\mathbb{T}$. The intersection $\bigcap \mathbf{J_n}$ contains an element t_0 with $t^{z_n} \in I_n$. $\square$

Lacunary sequences constitute an important source of examples of I_0-sets. Another source is of course formed by independent sets. Say that a subset $A = \{a_i \colon i \in I\}$ of an Abelian group is independent if $\langle\, a_j \,\rangle \cap \langle\, a_i \colon i \neq j \,\rangle = \{e\}$, then A is an I_0-set, for every function on $f \colon A \to \mathbb{T}$ extends to a homomorphism on $\bar{f} \colon \langle\, A \,\rangle \to \mathbb{T}$ and then to a character of G. Similarly, any set A such that the subgroup $\langle\, A \,\rangle$ is fre, is also an I_0-set. Of course no such set can be found in $\mathbb{Z}$ where any two elements are always dependent.

We begin now to draw the differences between I_0-sets and Sidon sets. It is interesting to observe how the joint continuity of multiplication in $G^{\mathcal{AP}}$ plays here a prominent role.

Example 3.5. *If $A = A_1 \cup A_2$ with $A_1 = \{6^n \colon n \in \mathbb{N}\}$ and $A_2 = \{6^n + n \colon n \in \mathbb{N}\}$, then A is a Sidon set but it is not an I_0-set.*

Proof. Both A_1 and A_2 are lacunary sets, hence I_0-sets and Sidon ssets and there is a quite deep theorem, due to Drury [**Dru70**], stating that the union of two Sidon sets is always a Sidon set (for this particular case easier proofs can be found). For a proof of the union theorem for Sidon sets the reader may consult section 5.5 of the monograph [**DR71**].

It is also a general fact that an I_0 set A set cannot be partitioned in two sets A_1 and A_2 in such a way that $A_1 - A_2 \cup A_2 - A_1$ contains an infinite symmetric set B (in this case $B = \{n \in \mathbb{Z} \colon n \neq 0\}$), let us see why: If α is an accumulation point of B in $G^{\mathcal{AP}}$, then 0 is an accumulation point of B, simply take a net b_η going to α and consider for every η some $\alpha(\eta) \geq b_\eta$ with $b_{\alpha(\eta)} \neq b_\eta$. Then the net $x_\eta = b_\eta - b_{\alpha(\eta)} \in B$ and converges to 0. It follows that either $0 \in \overline{A_1 - A_2}^{G^{\mathcal{AP}}} = \overline{A_1}^{G^{\mathcal{AP}}} - \overline{A_2}^{G^{\mathcal{AP}}}$ or $0 \in \overline{A_2 - A_1}^{Gap} = \overline{A_2}^{G^{\mathcal{AP}}} - \overline{A_1}^{G^{\mathcal{AP}}}$, and in either case we deduce that $\overline{A_1}^{G^{\mathcal{AP}}} \cap \overline{A_2}^{G^{\mathcal{AP}}} \neq \emptyset$. Since two disjoint subsets of A have nondisjoint closures in $G^{\mathcal{AP}}$, we conclude A is not an I_0-set. $\square$

Much is known about how a Sidon set is like, mainly for the group $\mathbb{Z}$, see the reference [**LR75**]. The series of papers [**GH06a, GHK06, GH05, GH06b**] gives a modern view on the lacunarity problems. Sidon sets on noncommutative groups are studied in [**Pic73, Cho82, Cho90**].

Concerning the description of Sidon sets and $\mathcal{WAP}$-interpolation sets the best information available seems to be yet the one obtained by Ruppert [**Rup85**]. This is a characterization of what we could call *uniformly approximable* $\mathcal{X}$-interpolation sets.

Definition 3.6. A subset $A \subset G$ of a topological group G is a *uniformly approximable* $\mathcal{X}$-interpolation set for some algebra $\mathcal{X} \subset \ell_\infty(G)$ if A is an $\mathcal{X}$-interpolation set and the characteristic function $1_A \in \mathcal{X}$.

Theorem 3.7. *Let G be a discrete group.*

(1) *(*[**Dru70**]*) If G Abelian, every Sidon set is uniformly approximable.*
(2) *(*[**Rup85**, Theorem 7]*) A subset $A \subset G$ is a uniformly approximable $\mathcal{WAP}$-interpolation set if and only if every infinite subset $B \subset A$ contains a finite subset F such that both*

$$\bigcap_{b\in F}\{x\colon bx \in A\} \quad \text{and} \quad \bigcap_{b\in F}\{x\colon xb \in A\} \tag{1.1}$$

are finite.
(3) *No infinite subset A of G is a uniformly approximable I_0-set.*

Proof. (1) This fact (the key step in the union theorem for Sidon sets) is too hard to be proved here. It requires a full comprehension of Sidon sets as "almost linearly independent". One fundamental tool are Riesz products, that are build as products of functions into $\mathbb{T}$ with the help of a different norm available on $B(G)$, the one it gets as the conjugate Banach space of $C^*(G)$, the group C^*-algebra. In the locally compact Abelian case, $B(G)$ with this norm is identified with $M(\widehat{G})$ via the Fourier-Stieltjes transform.

(2) If te condition (1.1) in Statement (2) does not hold, then it is possible to find two sequences (x_n) and (y_n) such that

(a) $y_n \in B$ for all n (b) $x_i y_n \notin \{x_j y_k \colon k \le j < n\}$ for all $i < n$,

(c) $x_n y_i \in A$ for $n \ge i$, and (d) $x_n y_i \notin \{x_k y_j \colon k < j \le n\}$ for all $i \le n$.

Then the set $C = \{x_n y_m \colon n \ge m\}$ is contained in A, but $0 = \lim_n \lim_m 1_B(x_n y_m) \neq \lim_m \lim_n f(x_n y_m) = 1$ which shows that B cannot be uniformly approximable. This proves necessity in (2). Sufficiency is obtained through another interesting property: *A is a uniformly approximable $\mathcal{WAP}(G)$-interpolation* *set if and only if* $\operatorname{cl}_{G^{\mathcal{WAP}}} A$ *is open and*

$$\operatorname{cl}_{G^{\mathcal{WAP}}} A \cap (G^{\mathcal{WAP}} \setminus G)^2 = \emptyset.$$

(3) Suppose $1_A \in \mathcal{AP}(G)$. The set $\{L_x 1_A \colon x \in G\}$ must be relatively compact in $\ell_\infty(G)$. Since $L_x 1_A = 1_{x^{-1}A}$, and $\|1_A - 1_B\|_\infty = 1$ if $A \neq B$, we conclude that there is an infinite family $B = \{x_{n(i)} \colon i \in I\}$ with $x_{n(i)}^{-1} A = x_{n(j)}^{-1} A$ for every $i, j \in I$. Obviously no finite subset of B can satisfy (1.1), hence A is not a uniformly approximable $\mathcal{WAP}$-interpolation set, let alone a uniformly approximable I_0-set. $\square$

Much more about Sidon sets in discrete groups is known, see for instance [**LR75, Cho82, Cho90**] and references therein.

In the case of $\mathcal{WAP}(G)$-interpolation sets, there is not so much information available, the following extension of statement (2) in Theorem 3.7 is worth mentioning. The theorem is due to Filali [**Fil07**] and has its roots in [**BF02**]. A similar approach also producing $\mathcal{WAP}$-interpolation sets can be found in [**FS04**, Theorem 1.4].

THEOREM 3.8. *Let $A = (x_n)$ be a sequence in a topological group and let U denote a compact symmetric neighbourhood of the identity in G. If A satisfies the following conditions:*

1. *For any neighbourhood V of the identity, $\bigcap_n \left(x_n^{-1} V x_n \cap x_n V x_n^{-1}\right)$ is again a neighbourhood of the identity.*
2. *$U^2 x_n \cap U^2 x_m = \emptyset$, for every $n \neq m$.*
3. *If $x \notin V^2$,*

$$xUT \cap UT \qquad \text{and} \qquad UTx \cap UT$$

are both relatively compact.

Then A is a $\mathcal{WAP}$-interpolation set.

PROOF (SKETCH). Let $f \colon A \to \mathbb{C}$ be bounded and let $g \colon G \to \mathbb{C}$ be continuous, positive and supported in U. Define $\bar{f} = \sum_n h(x_n) R_{x_n^{-1}} h$, that is

$$\bar{f}(s) = \sum_n h(x_n) g(s x_n^{-1}).$$

Clearly $\bar{f}$ is an extension of f. $\bar{f} \in \mathcal{LUC}(G)$ for it consists in a linear combination of $\mathcal{LUC}(G)$-continuous functions with disjoint supports (the support of $R_{x_n^{-1}}$ is contained in Ux_n). With the aid of property (1) one shows that $\bar{f} \in \mathcal{RUC}(G)$ and then Grothendieck's criterion shows that $\bar{f} \in \mathcal{WAP}(G)$. See [**Fil07**, Theorem 1] for full details. $\square$

2. Interpolation sets and ℓ_1-basis

The sole existence of an infinite interpolation set immediately implies that the cardinality of the compactification is at least $2^{\mathfrak{c}}$, as big as possible for separable groups. With the aid of Bourgain-Fremlin-Talagrand's dichotomy, good classifications of groups having some infinite $\mathcal{X}$-interpolations are available. This approach through Bourgain-Fremlin-Talagrand's dichotomy was first used in [**GH04**] and has been also used in the more general setting of dynamical systems (see for instance [**Gla07**]).

For this approach we need the connection between the concept of interpolation set and the concept of ℓ_1-basis.

Definition 3.9. A sequence $(a_n)_{n<\omega}$ contained in a Banach space B is called an ℓ^1-basis when there is some $\delta > 0$ such that

$$\left\|\sum_{n=1}^{m} \lambda_n a_n\right\| \geq \delta \sum_{n=1}^{m} |\lambda_n|, \qquad \text{for any complex numbers } \lambda_1, \ldots, \lambda_m.$$

If (a_n) is an ℓ^1-basis the correspondence sending each a_n to the canonical e_n vector of ℓ^1 (that with 0's everywhere save the 1 placed in the n-th. position) defines a topological isomorphism from the Banach subspace $\overline{\operatorname{sp}(\{a_n : n < \omega\})}$ of B generated by $\{a_n : n < \omega\}$ onto ℓ^1.

The connection between between interpolation sets and ℓ^1-basis is hinted below, it exploits the fact that ℓ_1-basis in $\mathcal{X}^*$ are actually $\mathcal{X}^{**}$.

Lemma 3.10. *Let G be a topological group and $\mathcal{X}$ a closed separating subalgebra of $\mathcal{CB}(G)$. If a countable subset $A = \{a_n : n \in \mathbb{N}\} \subseteq G$ is an $\mathcal{X}$-interpolation set, then A is an ℓ^1-basis in $\mathcal{X}^*$.*

Proof. We first remark the well-known fact that, as a consequence of the open mapping theorem, for a given $\mathcal{X}$-interpolation A there is a constant κ_A such that $\|\bar{f}\| \leq \kappa_A \|f\|_\infty$ for every bounded $f\colon A \to \mathbb{C}$ with extension $\bar{f} \in \mathcal{X}$.

Assume A is an $\mathcal{X}$-interpolation set. Let $\lambda_1, \ldots, \lambda_n$ be complex numbers and define $f\colon A \to \mathbb{C}$ by $f(a_k) = \dfrac{|\lambda_k|}{\lambda_k}$ for $1 \leq k \leq n$ and $f(a_k) = 0$ if $k > n$. This function f is the restriction to A of some $\bar{f} \in \mathcal{X}$ with $\|\bar{f}\| \leq \kappa_A$. Now,

$$\left\|\sum_{k=1}^{n} \lambda_k a_k\right\|_{\mathcal{X}^*} \geq \frac{1}{\kappa_A}\left|\bar{f}\left(\sum_{k=1}^{n} \lambda_k a_k\right)\right| = \frac{1}{\kappa_A}\left|\sum_{k=1}^{n} \lambda_k f(a_k)\right| = \frac{1}{\kappa_A}\sum_{k=1}^{n} |\lambda_k|.$$

□

REMARK 3.11. It is interesting to note that the converse of the above theorem is not true, if A is an ℓ^1-basis in $\mathcal{X}^*$, then every bounded function on A extends to a continuous functional on $\mathcal{X}^*$, i.e., to an element of the second conjugate $\mathcal{X}^{*\,*}$. The restriction of this function to $G^{\mathcal{X}}$ must happen to be $\sigma_{\mathcal{X}}(\mathcal{X}^*, \mathcal{X})$-continuous to deduce that $f \in \mathcal{X}$, and therefore that A is an $\mathcal{X}$-interpolation set. This nonetheless happens in some situations, for instance when $\mathcal{X}$ is the conjugate space of some Banach space B in such a way that there is an embedding $G \hookrightarrow B \hookrightarrow B^{**} = \mathcal{X}^*$ compatible with the embedding $G \hookrightarrow \mathcal{X}^*$. An example of paramount importance is the case of Sidon sets of discrete groups. A discrete group embeds in its group algebra $C^*(G)$ and the latter is a predual of $B(G)$.

COROLLARY 3.12. *Let G denote a countable discrete group. A sequence S in G is a Sidon set if and only if it is an ℓ_1-basis in $C^*(G)$ (or in $\mathcal{B}(G)^*$).*

This connection between interpolation sets and ℓ^1-basis gives the key for a qualitative criterion for their existence. It is founded on the celebrated Rosenthal's ℓ^1-theorem (already presented in Theorem 3.13.

THEOREM 3.13 (Rosenthal). *If (b_n) is a bounded sequence in a Banach space B, then either (b_n) has a weakly Cauchy subsequence (weakly$\equiv$ $\sigma(B, B^*)$) or a subsequence $(b_{n(k)})$ of (b_n) is an ℓ^1-basis.*

Theorem 3.13 gives a serious hint on how Banach spaces without ℓ^1-basis should look like. Rosenthal [**Ros77**] made it more precise and subsequently Fremlin, Bourgain and Talagrand [**BFT78**] gave the definitive characterization.

THEOREM 3.14 (Bourgain, Fremlin, and Talagrand [**BFT78**]). *Let L denote a separable and metrizable space and let F be a subset of $\mathcal{B}_1(L)$. If every countable subset of F has an accumulation point in $\mathcal{B}_1(L)$ then F has the following properties:*

(1) *F is relatively sequentially compact (i.e. every sequence has a subsequence convergent in $\mathbb{C}^F$).*
(2) *F has a compact closure in $\mathcal{B}_1(L)$.*
(3) *F is sequentially dense in a compact subset of $\mathcal{B}_1(L)$ (i.e. every point of $\mathcal{B}_1(L)$ is the pointwise limit of some sequence in F).*

Compact spaces that can be embedded in $\mathcal{B}_1(L)$ for some separable and metrizable L are usually named *Rosenthal-compact* spaces. By theorem 3.14

we have that a Rosenthal-compact space is necessarily sequentially compact, and sequentially separable.

In the usual terminology $\mathcal{B}_1(L)$ will be the subset of $\mathbb{R}^L$ consisting of limits of sequences of continuous functions. We always regard $\mathcal{B}_1(L)$ with the topology inherited from $\mathbb{R}^L$.

Actually, Rosenthal's theorem in the form of Theorem 3.13 is not directly applicable to all kinds of interpolation sets we are interested in, but its proof (an utterly useful application of Ramsey that works mainly with spaces of functions) does prove to be meaningful for the study of interpolation sets. We will use Rosenthal's theorem trough Todorcevic's nontrivial reformulation that encompasses as well Bourgain-Fremlin-Talagrand's theorem.

Theorem 3.15 ([**Tod97**]). *Suppose X is separable and completely metrizable (i.e. Polish) and let $\{f_n \colon n \in \mathbb{N}\} \subset CB(X, \mathbb{C})$ be a pointwise bounded sequence. Then either:*

(1) *There is a subsequence $\{f_{n(k)} \colon k \in \mathbb{N}\}$ such that $\mathrm{cl}_{\mathbb{C}^X}\{f_{n(k)} \colon k \in \mathbb{N}\}$ is (homeomorphic to) $\beta\mathbb{N}$, or*
(2) *Every point of $\mathrm{cl}_{\mathbb{C}^X}\{f_n \colon k \in \mathbb{N}\}$ is the limit of a subsequence of $\{f_n \colon n \in \mathbb{N}\}$*

Proof. This follows directly from from Todorcevic's Propositions 1 and 2 of Section 13 in [**Tod97**] in the case of real-valued functions. By Proposition 1 of loc. cit. (that uses Baire's characterization of (non-) class-1 functions) it is proved, that either $\mathrm{cl}_{\mathbb{R}^X}\{f_n \colon n \in \mathbb{N}\}$ is contained in $\mathcal{B}_1(X)$ the set of all Baire class-1 functions or $\{f_n \colon n \in \mathbb{N}\}$ has a subsequence $\{f_{n(k)} \colon k \in \mathbb{N}\}$ with $\mathrm{cl}_{\mathbb{R}^X}\{f_{n(k)} \colon k \in \mathbb{N}\} = \beta\mathbb{N}$. In the former case Proposition 2 of loc. cit., that actually corresponds to Bourgain-Fremlin-Talgrand's theorem, shows that every point in $\mathrm{cl}_{\mathbb{R}^X}\{f_n \colon n \in \mathbb{N}\}$ is the limit of some Cauchy subsequence. □

Exercise 16. Extend the preceding proof to the case of complex-valued functions.

Corollary 3.16. *Suppose X is separable and completely metrizable (i.e. Polish) and consider a pointwise bounded sequence $\{f_n \colon n \in \mathbb{N}\} \subset CB(X, \mathbb{R})$ with no pointwise Cauchy subsequence (i.e., has no Cauchy subsequences as a subset of $\mathbb{R}^X$). Then $\{f_n \colon n \in \mathbb{N}\}$ has a subsequence $\{f_{n(k)} \colon k \in \mathbb{N}\}$ with $\mathrm{cl}_{\mathbb{C}^X}\{f_{n(k)} \colon k \in \mathbb{N}\}$ homeomorphic to $\beta\mathbb{N}$.*

In the following subsections we examine the comsequences of Theorem 3.14 and Theorem 3.16 to the existence and abundance of interpolation sets.

3. When is $|G^{\mathcal{X}}| < 2^{\mathfrak{c}}$? The Bourgain-Fremlin-Talagrand dichotomy

We now see how Bourgain-Fremlin-Talagrand's theorem applies to the problem of (non-)existence of $\mathcal{X}$-interpolation sets. We want to regard the elements of G as sets of continuous functions on the algebra $\mathcal{X}$. For Rosenthal-like theorems to apply we thus need a Polish topology on $\mathcal{X}$ that makes the elements of G continuous. This task can be difficult, or at least unnatural, if we insist in using the uniform topology on the whole $\mathcal{X}$. Typically we find a big enough subset of $\mathcal{X}$ that is Polish for some weaker topology.

Next theorem states what we need and its corollaries will show that the hypothesis are natural. We say here that a subset L of a Banach algebra is generating if the closed subalgebra generated by L is $\mathcal{X}$. Observe that a generating set necessarily separates the points of $G^{\mathcal{X}}$.

THEOREM 3.17. *Let G be a separable topological group and let $\mathcal{X}$ be a uniformly closed separating subalgebra of $\ell^\infty(G)$ that contains a generating set L such that (L,τ) is Polish for some topology finer than that of pointwise convergence. Then G has no $\mathcal{X}$-interpolation set if and only if the spectrum $G^{\mathcal{X}}$ is a Rosenthal-compact space.*

PROOF. Since L separates points of $G^{\mathcal{X}}$, we can regard $G^{\mathcal{X}}$ as a subset of $\mathbb{C}^L$ (that is, the restriction mapping from $\mathcal{X}$ to L will be a homeomorphism). Under this identification, each element of G corresponds to a continuous functions $(L,\tau) \to \mathbb{C}$.

If G has no infinite $\mathcal{X}$-interpolation set, no subset of A can have a closure in $\mathbb{C}^L$ that is homeomorphic to $\beta\mathbb{N}$ (statement (3) of lemma 3.3), and Theorem 3.15 shows that every infinite subset A of G contains an infinite $\mathbb{C}^L$-Cauchy sequence. We deduce therefore that G is relatively sequentially compact in $\mathbb{C}^L$. Theorem 3.14 applies now to show that G has a compact closure in $\mathcal{B}_1(L)$. Since $\mathrm{cl}_{\mathbb{C}^L} G = G^{\mathcal{X}}$ we conclude that $G^{\mathcal{X}} \subseteq B_1(L)$ and thus that $G^{\mathcal{X}}$ is Rosenthal-compact.

If conversely $G^{\mathcal{X}}$ is Rosenthal-compact, then $|\epsilon_{\mathcal{X}}(G)| \le \mathfrak{c}$ because Rosenthal-compact spaces are sequentially separable (again theorem 3.14) and statement (3) of lemma 3.3 shows that G cannot contain any $\mathcal{X}$-interpolation set (the cardinality of Stone-Čech compactifications of infinite discrete spaces is at least $2^{\mathfrak{c}}$). $\square$

COROLLARY 3.18. *A second countable locally compact group G has no infinite Sidon subsets if and only if $G^{\mathcal{B}}$ is Rosenthal compact.*

PROOF. Choose $L = \mathcal{P}_1(G)$ the set consisting of positive-definite functions f with $\|f\|_\infty = f(1_G) = 1$ and $\tau = \sigma(L^\infty(G), L^1(G))$. τ is a Polish topology and by Raikov's theorem (see Theorem 3.31 of [**Fol95**]) agrees with the compact open topology on $\mathcal{P}_1(G)$. Since $B(G)$ is spanned by L, Theorem 3.17 can be applied. $\square$

COROLLARY 3.19 ([**GH04**]). *If G is a second countable locally compact group, G has no infinite I_0-sets if and only if $G^{\mathcal{AP}}$ is Rosenthal compact.*

PROOF. We now choose $L = \bigcup_n \mathcal{P}_{1,n}(G)$ where $\mathcal{P}_{1,n}$ stands for the set of elements of $\mathcal{P}_1(G)$ whose associated GNS representation (see Proposition 1.6, (3)) has dimension n.

L is again a generating set for $AP(G)$. While $\mathcal{P}_{1,n}(G)$ is again Polish for the $\sigma(L^\infty(G), L^1(G))-$ (=compact-open) topology it is not clear at all whether L is. We choose instead the topological sum as the τ-topology, that surely satisfies this condition and apply Theorem 3.17. $\square$

Being Rosenthal compact is indeed a condition on $G^{\mathcal{X}}$ that imposes severe restrictions on $\mathcal{X}$ as is clearly manifested when $\mathcal{X} = AP(G)$:

THEOREM 3.20 ([**GH04**]). *Let G be a second countable topological group. The following assertions are equivalent.*

1. *G has no I_0-sets.*
2. *The Bohr compactification $G^{\mathcal{AP}}$ of G is metrizable.*
3. *$AP(G)$ is separable.*
4. *G has countably many inequivalent finite dimensional unitary representations.*

PROOF. (2), (3) and (4) are always equivalent, as the topological weight of a compact group equals the number of irreducible, inequivalent representations ([**HR63**, 28.51]) and $AP(G) = C(G^{\mathcal{AP}}, \mathbb{C})$.

By Corollary 3.19 G has no I_0-sets if and only if $G^{\mathcal{AP}}$ is Rosenthal compact, thus (2) implies (1). To see that (1) implies (2) we simply observe that Rosenthal-compact topological groups are metrizable (Rosenthal-compact sets have G_δ-points, see [**Tod97**]). $\square$

COROLLARY 3.21. *The Bohr compactification $G^{\mathcal{AP}}$ of a topological group G has cardinality $|G^{\mathcal{AP}}| \geq 2^{\mathfrak{c}}$ as soon as G has uncountably many inequivalent finite dimensional representations.*

The above Corollary applies for instance to all infinite locally compact Abelian groups and all groups admitting an infinite Abelian quotient.

The analog of Theorem 3.20 for Sidon sets or $\mathcal{WAP}$-interpolation set is not true, as there are groups with $\mathfrak{c}$-many inequivalent unitary representations (infinite dimensional) with small compactifications:

THEOREM 3.22. *If G is a simple Lie group, then every $\phi \in \mathcal{WAP}(G)$ goes to 0 at infinity. Thus $G^{\mathcal{WAP}} = G^*$, the one-point compactification of G.*

A self contained proof can be found in [**Rup84**, Chapter 5]. These groups are of course not SIN (otherwise we would enter in contradiction with Theorem 3.25). For SIN groups the size of $G^{\mathcal{B}}$ does not seem to be known; it is thus uncertain how the condition $G^{\mathcal{B}}$ Rosenthal compact affects the structure of G.

QUESTION 8. Is it true that $|G^{\mathcal{B}}| \geq 2^{\mathfrak{c}}$ for all discrete groups?

It is interesting to remark that a considerably weaker Conjecture of Chou [**Cho82**] remains open: *Every discrete group G has some $\phi \in B(G)$ that cannot be decomposed as $\phi = \phi_1 + \phi_2$ with $\phi_1 \in \mathcal{AP}(G)$ and $\phi_2 \in C_0(G)$.* The class of discrete groups admitting infinite Sidon sets, i.e., the groups with $|G^{\mathcal{B}}| \geq 2^{\mathfrak{c}}$ is nevertheless quite large.

THEOREM 3.23. *If H is a closed subgroup of a SIN group G, the Sidon sets in H are Sidon sets in G.*

The following is just a sample of the possible consequences of Theorem 3.23.

COROLLARY 3.24. *If a discrete group G has some Abelian or some free subgroup, then $|G| \geq 2^{\mathfrak{c}}$.*

PROOF OF THEOREM FOR DISCRETE G. This is easy because every positive definite function ϕ on H extends to a positive definite function $\bar{\phi}$ on G, namely to $\bar{\phi}(g) = \phi(g)$ if $g \in H$, and $\bar{\phi}(g) = 0$, if $g \notin H$, [**HR70**, 32.43]. The rest of the proof is completely straightforward.

The restriction mapping $R\colon B(G) \to B(H)$ is known to be surjective for several pairs (G, H). R is in particular always surjective G is SIN, see [**McM72**], or [**Kan04**] and the bibliography therein for more recent developments. □

We already know that Theorem 3.23 cannot be extended to all locally compact groups (e.g. simple Lie groups, whose weakly almost periodic functions vanish at infinity contain infinite discrete Abelian groups). We will also see that this same result is far from being true for I_0-sets. (even if G is discrete).

3.1. The case of $\mathcal{WAP}$-interpolation sets: complete answers. It is unclear whether the above theorems could work for $\mathcal{WAP}(G)$ but the fact is we do not need them. In general the main advantage of $B(G)$ (or even $\mathcal{B}(G)$) over $\mathcal{WAP}(G)$ is that the former can be seen as the dual of a C^*-algebra under a suitable norm. This helps for instance in finding a natural topology for Lemma 3.17 to hold. While the looser relation of $\mathcal{WAP}(G)$ with the analytic structure carried by topological groups could be blamed for the lack of theorems like 3.17, this same condition makes it easier to build $\mathcal{WAP}$-interpolation sets by hand:

THEOREM 3.25 ([**Cho69**] for the discrete case; [**Fil07**] and [**FS04**] for the general case). *Every subset of a locally compact SIN group G that is not precompact contains some infinite $\mathcal{WAP}$-interpolation set.*

SKETCH OF PROOF FOR THE DISCRETE CASE. Let $B \subset G$ be infinite. We construct a sequence $A = (a_n) \subset A$ inductively, taking care that

$$a_{n+1} \notin \left\{x_i \cdot x_j \cdot x_k \colon \text{ with } x_l \in \{a_l, a_l^{-1}\} \text{ and } 1 \leq i, j, k \leq n\right\} \tag{3.1}$$

The set A thus defined satisfies the condition of statement (2) (even a stronger one, it is a T-set) in Theorem 3.7 and therefore is a $\mathcal{WAP}$-interpolation set. $\square$

EXERCISE 17. Use the idea just applied to discrete groups to prove Theorem 3.25. Instead of applying Theorem 3.7, one should obviously apply Theorem 3.8. Recall that a locally compact group is SIN if it has a neighbourhood basis $\{U_i\}_{i\in I}$ at the identity consisting of invariant sets, i.e., with $U_i g = gU_i$ for all $g \in G$ and all $i \in I$.

4. When $\mathcal{X}$-interpolation sets are everywhere.

We have so far found some examples of locally compact groups containing no infinite I_0, Sidon or even infinite $\mathcal{WAP}$-interpolation sets at all, and thus some instances of groups with very simple $\mathcal{X}$-compactifications. We have however also seen that for large classes of groups (SIN groups, for instance) this is not the typical case. The structure of the compactifications in these cases is usually intricate as we will be seeing.

In this section we present a sample of how plentiful $\mathcal{X}$-interpolation sets may get to be. Example 3.4 already proved that every infinite subset B of $\mathbb{Z}$ must contain some infinite I_0-set (a lacunary sequence of ratio at least 4; as we said any lacunary set actually works) and Theorem 3.25 showed that every infinite subset of a discrete group contains some infinite $\mathcal{WAP}$-interpolation set. Going further in that direction one can prove:

THEOREM 3.26. [**GH99**, Lemma 2.3] *Let G be a locally connected group which is either locally compact or completely metrizable. A subset of $A \subset$* $\mathrm{Hom}(G, \mathbb{T})$ *is either equicontinuous (and thus relatively compact in the topology of uniform convergence on compact sets), or contains an infinite ε-Kronecker (ε arbitrary).*

Here, we have used a lacunary-type property stronger than being I_0: say that $A \subset G$ is ε-Kronecker if for every $f\colon A \to \mathbb{T}$ there is some $\chi\colon G \to \mathbb{T}$ such that $|f(x) - \chi(x)| < \varepsilon$ for all $x \in A$. For the many lacunary properties of ε-Kronecker sets, see the series of papers [**GH06a, GHK06, GH05, GH06b**]. Of course, ε-Kronecker sets with $\varepsilon < \sqrt{2}$ are I_0-sets.

COROLLARY 3.27. *Every infinite subset of a finitely generated Abelian group contains an infinite I_0-set. Every unbounded subset of $\mathbb{R}^n$ contains an infinite I_0-set.*

If G is a discrete subset and $A \subseteq G$ is infinite we may fix two disjoint intervals as in Example 3.4 and define for every $C \subset A$ and every $\phi : C \to \{1, 2\}$ a set of characters like this:

$$N(\phi, C) = \{\chi \in \widehat{G} : \chi(c) \in I_{\phi(c)} \text{ for all } c \in C \}.$$

The elements of the family

$$\mathcal{X} = \{C \subseteq A : \quad N(\phi, C) \neq \emptyset \text{ for all } \phi \in \{1, -1\}^C\}$$

are clearly I_0-sets. The family $\mathcal{X}$ can be ordered in such a way that for any of its maximal elements B, it follows that $A \subseteq \langle B \rangle$ so that either A is contained in a finitely generated subgroup (and Lemma 3.26 applies) or B is an infinite I_0-subset of A with $|B| = |A|$. Putting everything together we obtain:

THEOREM 3.28. [**GH99**] *A subset A of an LCA group G always contains an I_0-set of cardinality $\kappa(\bar{A})$, the compact-covering of $\bar{A}$ the closure in G of A.*

The scope of Lemma 3.26 is not limited to locally compact groups, we have for instance:

COROLLARY 3.29. *The following sets contain an infinite I_0-set:*

(1) *Sets with noncompact closure in free Abelian topological groups and strict inductive limits of locally compact groups.*
(2) *Unbounded subsets of locally convex spaces.*
(3) *Sets with noncompact closure in locally convex spaces with Schur's property (such as ℓ_1).*
(4) *Sets with noncompact closure in nuclear groups.*

The first two statements appear in [**GH99**] , the third in [**HGM99**] and the fourth in [**BMP96**].

4.1. Applications of Rosenthal ℓ_1-theorem. A fundamental tool for determining the existence of $\mathcal{X}$-interpolation sets is obtained by relating their presence to the absence of weakly convergent sequences. This is obtained via Rosenthal-type theorems that use 3.15 like the following one:

THEOREM 3.30. *Let G be a separable and metrizable locally compact group and $\mathcal{X}$ a subalgebra of $\ell^\infty(G)$ as in Theorem 3.17. Any sequence $\{g_n : n \in \mathbb{N}\}$ contained in G has a subsequence $\{g_{n(k)} : n \in \mathbb{N}\}$ with one of the following two properties:*

(1) *$\{g_{n(k)} : k < \omega\}$ is an $\mathcal{X}$-interpolation set.*
(2) *$\{g_{n(k)} : k < \omega\}$ is a pointwise Cauchy sequence.*

Translating theorem 3.30 to concrete algebras one gets (see [**GH04**]):

THEOREM 3.31. *Let G be a separable and metrizable locally compact group. Any sequence $\{g_n : n \in \mathbb{N}\}$ contained in G has a subsequence $\{g_{n(k)} : n \in \mathbb{N}\}$ with one of the following two properties:*

(1) *$\{g_{n(k)} : k < \omega\}$ is a Sidon set (I_0-set).*
(2) *$\{g_{n(k)} : k < \omega\}$ is a Cauchy sequence for the topology of pointwise convergence on $\mathcal{B}(G)$ (resp. $\mathcal{AP}(G)$).*

The separability hypothesis can be removed in Theorem 3.31, see [**GH04**] for details. On the contrary, the metrizability hypothesis cannot be disposed of. For a (compact) counterexample, see [**GH04**].

It is usually much harder to find I_0-sets in noncommutative groups. One reason for this fact is that no analog of Lemma 3.23 holds. The paper [**Her08**] contains examples of how dramatically this can fail for noncommutative discrete groups. The abundance of I_0-sets is actually a rather commutative property [1]. Actually no group having infinite I_0-sets contained in

[1]Or maybe a property that requires a high degree of independence

every infinite subset is known, apart of Abelian by finite groups. For some classes of groups it is known that this is indeed the only possible case.

THEOREM 3.32 (Theorem 4.1 and Corollary 6.3 of [**Her08**] and [**WR96**], respectively). *Suppose G is either (a) a finitely generated group G without non-abelian free subgroups or (b) an FC-group*[2]. *If every infinite subset of G contains an infinite I_0-set, then G is abelian by finite.*

SKETCH OF PROOF. In case (a) Hernández shows that G has sequences $S = \{x_n\}$ that converge to 1_G in the topology of $G^{\mathcal{AP}}$. Since $\mathrm{cl}_{G^{\mathcal{AP}}} S = S \cup \{e\}$ is a countable compact set and $\mathrm{cl}_{G^{\mathcal{AP}}} A \subset S$ for every $A \subset S$, no such A can be an I_0-set.

In case (b), Riggins and Wu show that only countably many finite-dimensional representations of the commutator subgroup G' of an FC group G are inequivalent. The topology that G' inherits from $G^{\mathcal{AP}}$ is therefore metrizable and produces many nontrivial convergent sequences unless it is finite. □

See [**GHW07**] for a summary on what is known about the existence of I_0-sets in noncommutative locally compact groups. For $\mathcal{B}(G)$ the situation is quite different and Sidon sets are more abundant. De Michele and Soardi [**DMS76**] show for instance that discrete FC-groups have infinite Sidon sets in every infinite subset, this trivially generalizes to locally compact [FC] groups (groups with precompact conjugacy classes):

THEOREM 3.33. *If G is a locally compact [FC] group, and $A \subset G$ is not precompact, then A contains an infinite Sidon subset.*

PROOF (SKETCH). If G is a discrete FC group, $G/Z(G)$ is topologically isomorphic to a direct sum of finite groups. This gives great control on positive definite functions and makes possible to associate to every sequence (y_n) in G a positive definite function with $\phi(y_n)$ not convergent. Theorem 3.31 then proves the discrete case. The general case follows easily from the structure of $[FC]$ groups: every $[FC]$ group is the extension of a compact group by the product of a vector group and a discrete SIN group, see for instance [**Liu73**]. □

[2] An FC group is a discrete group all whose conjugacy classes are finite. A locally compact group whose conjugacy classes are relatively compact is said to be an [FC] group.

4

Algebraic structure of compactifications

We have already had a glimpse over the algebraic structure of compactifications in Section 1. We know for instance that all four compactifications, $G^{\mathcal{LUC}}$, $G^{\mathcal{WAP}}$, $G^{\mathcal{B}}$ and $G^{\mathcal{AP}}$ admit a binary operation that extends the multiplication of G and makes them into a semigroup. We also know that only $G^{\mathcal{AP}}$ has a group structure. It is also important to recall that this operation is jointly continuous only for $G^{\mathcal{AP}}$; $G^{\mathcal{WAP}}$ and $G^{\mathcal{B}}$ have separately continuous multiplication while only right multiplication is continuous for $G^{\mathcal{LUC}}$.

In this section we will deal almost exclusively with Abelian groups, very often discrete, since as we will see the algebraic structure of the compactifications is not well understood even in these cases. We also recall that for Abelian G, $G^{\mathcal{WAP}}$, $G^{\mathcal{B}}$ and $G^{\mathcal{AP}}$ are all commutative, while $\mathbb{Z}^{\mathcal{LUC}}$ is not. Concerning the algebraic structure we will only touch some basic facts as the existence of idempotents and cancellability properties of semigroup compactifications. The algebraic structure of compactifications is a rich subject with many connections with other ares, the monograph, [**HS98**] gives many examples and is devoted almost exclusively to $\mathbb{Z}^{\mathcal{LUC}}$! Some of the results presented here are part of a joint work in progress with M. Filali and appear here with his kind permission.

1. Idempotents

Idempotents are essential in the description (and some applications as those to Ramsey theory) of the algebraic structure of semigroup compactifications. Observe for instance how important are projections in C^*-algebras, including characteristic functions in $L_\infty(X,\mu)$-spaces.

Definition 4.1. Let S be a semigroup. The *set of idempotents* of S will be denoted as $E(S)$.

If S is commutative $E(s)$ is obviously a subsemigroup of S.

The set of idempotents is never empty in the semigroups we are interested in. This now well-known fact was first proved in this generality by Ellis, [**Ell69**].

THEOREM 4.2 (Ellis theorem). *Every compact right topological semigroup contains at least one idempotent.*

PROOF (SKETCH). Take a set A that is minimal in the family (ordered by inclusion) $\mathcal{A} = \{T \subset S \colon T \text{ closed, and } TT \subset T\}$.

If $x \in A$, $Ax \in \mathcal{A}$ and, since $Ax \subset A$, $Ax = A$. This shows that the set $B = \{a \in A \colon ax = x\}$ is nonempty. Again $BB \subset B$ and closed, thus $B = A$. It follows that $xx = x$. $\square$

LEMMA 4.3. *Suppose S and T are compact right topological semigroups and that $\phi \colon S \to T$ is a continuous surjective homomorphism. Then $\phi(E(S)) = E(T)$.*

PROOF. For every $e \in E(T)$, the set $V_e = \phi^{-1}(\{e\})$ is a compact semitopological semigroup. By Ellis Theorem there must be an idempotent in V_e and hence $e \in \phi(E(s))$ and $E(T) \subset \phi(E(S))$. The other inclusion is trivially checked. $\square$

THEOREM 4.4. *$\mathbb{Z}^{\mathcal{LUC}}$, $\mathbb{Z}^{\mathcal{WAP}}$ and $\mathbb{Z}^{\mathcal{B}}$ have uncountably many different idempotents.*

PROOF. Let B denote the unit ball of the space $L_\infty(X, \mu)$, that is the set of essentially bounded mappings $f \colon X \to \mathbb{C}$ with $|f| \leq 1$ a.e. If equipped with the $\sigma(L_\infty, L_1)$-topology B is semitopological semigroup. For every measurable subset $A \subset X$ the characteristic function $1_A \in E(B)$. We will assume that μ is a measure defined on a Kronecker subset X of $\mathbb{T}$. Hence every function $f \colon X \to \mathbb{T}$ can be approximated by characters.

Identifying every element of $f \in L_\infty(X, \mu)$ with the *multiplication operator* M_f on $L_2(G)$, $L_\infty(X, \mu)$ is a commutative von Neumann algebra (*the only* one up to isomorphism, actually) and, B turns into a semigroup of operators under multiplication. As such the $\sigma(L_\infty, L_1)$-topology, turns into the weak operator topology (see, for instance, Proposition 4.50 of [**Dou98**]).

Let $L_0(X, \mu, \mathbb{T})$ denote the subset of B consisting of functions with $|f| = 1$, a.e. with the $\sigma(L_\infty, L_1)$-topology (that here coincides with the topology of convergence in measure, thence the subindex 0). This is the unitary group of the von Neumann algebra $L_\infty(X, \mu)$ and is well known to be weak operator dense in B.

Define now $f_0 \in L_\infty(X, \mu, \mathbb{T})$ as $f_0(e^{it}) = e^{it}$. We are actually defining a unitary representation $\pi \colon \mathbb{Z} \to L_\infty(X, \mu, \mathbb{T}) \subset \mathcal{U}(L_2(X, \mu))$ of $\mathbb{Z}$ on $L_2(X, \mu)$, as $\pi(n) = f_0^n$. For each $h \in L_2(X, \mu)$ the functions $P_f \colon \mathbb{Z} \to \mathbb{C}$ given by $P_f(n) = \int f_0^n(x)|h(x)|^2 \, d\mu$ are positive definite (it is a diagonal coefficient of π) and as thus extend to $\bar{P}_f\colon \mathbb{Z}^{\mathcal{B}} \to \mathbb{C}$. We have therefore that the map π extends to a continuous homomorphism $\bar{\pi}\colon G^{\mathcal{B}} \to B$. The fact that X is a Kronecker set is used at this point to observe that:

$$\overline{\{f_0^n\colon n\in\mathbb{Z}\}}^{\mathrm{WOT}} = \overline{L_\infty(X,\mu,\mathbb{T})} = B,$$

and, hence, that $\bar{\pi}$ is onto.

Since X has uncountably many different measurable sets, B has that many idempotents and it follows from Lemma 4.3 that so will do $\mathbb{Z}^{\mathcal{B}}$, $\mathbb{Z}^{\mathcal{WAP}}$ and $\mathbb{Z}^{\mathcal{LUC}}$. □

The actual number of idempotents in $\mathbb{Z}^{\mathcal{WAP}}$ is $2^{\mathfrak{c}}$, [**BM72**]

The question of whether $E(\mathbb{Z}^{\mathcal{WAP}})$ is closed or not was one of the important questions regarding compact semitopological semigroups. It was solved independently by [**BP00**] and [**BLM01**]. Theorem 4.4 shows that the answer to that question is negative. The argument of Theorem 4.4 has been taken from [**BLM01**] and was devised to solve this latter question. We have benefited from the helpful exposition included in [**Sal03**].

Corollary 4.5 ([**BP00**] and [**BLM01**]). *The idempotent semigroup $E(\mathbb{Z}^{\mathcal{WAP}})$ is not closed.*

Proof. We use the proof of Lemma 4.4 and its notation. It is enough to find a sequence of subsets $A_n \in \mathcal{X}$ such that 1_{A_n} does not converge to a characteristic function for the $\sigma(L_\infty, L_1)$-topology. We can take

$$A_n = \bigcup_{k=0}^{2^n-1} \left[\frac{k}{2^n}, \frac{k}{2^n} + \frac{1}{2^{n+1}}\right),$$

then 1_{A_n} converges to the constant function $1/2$.

This shows that $E(B)$ is not closed, since $E(\mathbb{Z}^{\mathcal{WAP}})$ is mapped onto $E(B)$ by Lemma 4.3, we see that the latter cannot be closed. □

2. Cancellability

Another set of properties that give information about the algebraic structure of a semigroup are those related to cancellability. Cancellation is never possible in $G^{\mathcal{B}}$, $G^{\mathcal{WAP}}$ or $G^{\mathcal{LUC}}$ as they contain many idempotents: let S denote any of these semigroups if $1_G \neq f \in E(S)$, then $f\cdot f = f\cdot 1_G$.

Even if cancellation is never possible in these compactifications one can hope for weaker forms of cancellability:

Questions 4.6. Let $G^{\mathcal{X}}$ denote one of the compactifications: $G^{\mathcal{B}}$, $G^{\mathcal{WAP}}$ or $G^{\mathcal{LUC}}$

(1) Does $G^{\mathcal{X}}$ contain any *cancellable elements*?, i.e. is there any $p \in G^{\mathcal{X}}$ such that whenever $q, s \in G^{\mathcal{X}}$ are such that $q \neq s$, then $pq \neq ps$?
(2) Does $G^{\mathcal{X}}$ contain elements such that $xg \neq x$ and $gx \neq x$ for all $g \in G$ with $g = 1_G$.
(3) Are there other *bigger* sets U in $G^{\mathcal{X}}$ such that $up \neq uq$ whenever $u \in U$ and $p, q \in G^{\mathcal{X}}$?
(4) Is the action of G by multiplication on $G^{\mathcal{LUC}}$ effective? That is to say, given $1_G \neq g \in G$, is there $p_g \in G^{\mathcal{X}}$ such that $p_g g \neq p_g$?

We can say here that the strongest answers for the above Questions have been obtained for $G^{\mathcal{LUC}}$. That in spite of being, at least in principle, the most complicated of these compactifications. We summarize here some of the known answers:

(1) $\mathbb{Z}^{\mathcal{LUC}}$ and, in general, $G^{\mathcal{LUC}}$ when G has uncountably many representations, has cancellable elements, see Theorem 4.11. It is not known whether $\mathbb{Z}^{\mathcal{WAP}}$ has any cancellable elements.
(2) The answer to (2) is positive for $G^{\mathcal{LUC}}$, for is locally compact G. In this case the action of G on $G^{\mathcal{LUC}}$ is *free*, i.e., $pg \neq p$ for any $g \neq 1_G$ and $p \in G^{\mathcal{LUC}}$, see Theorem 4.7. This result is known as Veech's theorem. $G^{\mathcal{WAP}}$ also satisfies Veech's theorem when G is maximally almost periodic (this is rather trivial). On the opposite side Veech theorem must fail in those locally compact groups with $\mathcal{WAP}(G) \subset C_0(G)$.
(3) This can be done in some cases, see Theorem 4.16 and Theorem 4.17
(4) It should be mentioned that all these cancellation properties fail completely in some nonlocally compact massive groups, like so-called *extremely amenable* groups. If G is such a group, there is $p \in G^{\mathcal{LUC}}$ with $gp = p$ for every $g \in G$ and the answer to questions (2), (3) and (4) is negative. The theory of extremely amenable groups has been recently developed by Pestov, see [**Pes07a**] for an account. The groups $\mathcal{U}(\mathbb{H})$ and $L_\infty(X, \mu, \mathbb{T})$ are examples of extremely amenable groups.

We begin to review some of the available answers to the questions in 4.6 with the well-known Veech's theorem.

THEOREM 4.7 (Ellis [**Ell60**]for discrete G, Veech for locally compact G[**Vee77**]). *Let G be a locally compact group. If $p \in G^{\mathcal{LUC}}$, $g \in G$ and $g \neq 1_G$, then $pg \neq p$. In other words, the action by multiplication of G on $G^{\mathcal{LUC}}$ is free.*

PROOF (SKETCH). We sketch Pym's proof of Veech's theorem, see [**Pym99**] for full details.

First, there is a partition of G into three sets A_0, A_1 and A_2 in such a way that $R_g(A_i) \cap A_i = \emptyset$ for $i = 1, 2, 3$. It suffices to take A_1 maximal with respect to the property $R_g(A_1) \cap A_1 = \emptyset$ (such sets exists because $gh \neq h$); then $A_0 = R_g(A_1)$ and $A_2 = G \setminus (A_1 \cup A_0)$.

If G is discrete, disjoint subsets of G have disjoint closures in $G^{\mathcal{LUC}} = \beta G$ and given $s \in G^{\mathcal{LUC}}$, p belongs to the closure of precisely one of the sets A_i. It is then easy to see that $pg \neq p$.

The general case can be derived through Pym's Local Structure Theorem for $G^{\mathcal{LUC}}$. It roughly states that every element $p \in G^{\mathcal{LUC}}$ is in the closure of some $\mathcal{LUC}$-interpolation set $\mathcal{X}$. Since $\mathrm{cl}_{G^{\mathcal{LUC}}}\mathcal{X}$ is homeomorphic to $\beta\mathcal{X}$ and G is open in $G^{\mathcal{LUC}}$ (G is locally compact), an argument similar to the one used above for discrete groups works. $\square$

The proof of Lemma 4.7 gives the first proof of how useful interpolation sets are in dealing with the algebraic structure of G. We will next see some more instances.

THEOREM 4.8. *Let G be a discrete Abelian group and let $T \subset G$ be a uniformly approximable $\mathcal{WAP}$-interpolation set. If $p \in \mathrm{cl}_{G^{\mathcal{WAP}}} T$ then $pq \neq p$ for any $1_G \neq q \in G^{\mathcal{WAP}}$*

PROOF. By [**Rup85**] all uniformly approximable $\mathcal{WAP}$-interpolation set(see the last sentence in Theorem 3.7) satisfy the property

$$\mathrm{cl}_{G^{\mathcal{WAP}}} T \cap (G^{\mathcal{WAP}} \setminus G)^2 = \emptyset.$$

If $q \notin G$, $pq \in (G^{\mathcal{WAP}} \setminus G)^2$, so $pq \neq p$. If $1_G \neq q \in G$, we consider a character $\chi \colon G \to \mathbb{T}$, with $\chi(q) \neq 1_{\mathbb{T}}$; $\chi \in \mathcal{WAP}(G)$ and admits an extension $\bar{\chi} \colon G^{\mathcal{WAP}} \to \mathbb{T}$. Since $\mathbb{T}$ is a group it follows from $\bar{\chi}(pq) = \bar{\chi}(p)$ that $\chi(q) = 1_{\mathbb{T}}$, a contradiction showing that $pq \neq p$. $\square$

The above theorem is essentially the construction of [**BF02**] and [**Fil07**]. In these references the construction is taken over to cover, at least, all locally compact SIN groups. This more or less the highest generality possible, examples of [**FS04**] and [**Fil07**] show that $\mathcal{WAP}$-interpolation sets are not so easy to construct in locally compact IN groups. Filali [**Fil07**] nevertheless manages to develop this technique to work in the class of E-groups introduced by Chou [**Cho75**].

We do not know at present if the above theorem holds for Sidon sets, but the proof of Lemma 8.33 of [**HS98**] also leads to similar results for this algebra, see Corollary 4.11 below.

If $\mathcal{X} \supset \mathcal{AP}(G)$ we use the symbol $b^{\mathcal{X}}\colon G^{\mathcal{X}} \to G^{\mathcal{AP}}$ to denote the canonical quotient semigroup homomorphism, recall that $b^{\mathcal{X}}_{|G} = b$, where $b = \epsilon_{\mathcal{AP}}$ is the Bohr map. Note that the the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "b"] \arrow[d, "\epsilon_{\mathcal{X}}"] & G^{\mathcal{AP}} \\ G^{\mathcal{X}} \arrow[ur, "b^{\mathcal{X}}"'] & \end{tikzcd}$$

LEMMA 4.9. *Let G be a maximally almost periodic topological group and let $\mathcal{X}$ be an algebra with $\mathcal{X} \supset \mathcal{AP}(G)$. Consider $S \subset G$ such that $b(S)$ is discrete, then for every $t \in S$ and $p \in \operatorname{cl}_{G^{\mathcal{X}}} S$, $b(t) = b^{\mathcal{X}}(p)$ implies $p = \epsilon_{\mathcal{X}}(t)$.*

PROOF. Let $p = \lim_j \epsilon_{\mathcal{X}}(s_j)$, with $s_j \in S$. Then

$$\lim_j b(s_j)) = \lim_j b^{\mathcal{X}}(\epsilon_{\mathcal{X}}(s_j)) = b^{\mathcal{X}}(p) = b(t).$$

And we deduce that $\lim_j b(s_j) = b(t)$. By hypothesis $b(A)$ is discrete in this topology and we must have that eventually $b(s_j) = b(t)$. Since b is injective, we obtain as a consequence that $s_j = t$ and hence that $p = \epsilon_{\mathcal{X}}(t)$. $\square$

THEOREM 4.10. *Let G be a maximally almost periodic group, let $\mathcal{X}$ be an algebra with $\mathcal{X} \supset \mathcal{AP}(G)$ and let $T \subset G$ satisfy the following two properties:*

(1) *T is discrete as a subset of $G^{\mathcal{AP}}$, i.e., if $b\colon G \to G^{\mathcal{AP}}$ is the canonical map, $b(T)$ is a discrete subset.*
(2) *The characteristic function $1_T \in \mathcal{X}$*

If $p \in \operatorname{cl}_{G^{\mathcal{X}}} T$ then $pq \neq p$ for any $1_G \neq q \in G^{\mathcal{X}}$.

PROOF. Let $q \in G^{\mathcal{X}}$ be such that $pq = p$. Let (t_i) be a net in T such that $\lim_i \epsilon_{\mathcal{X}}(t_i) = p$. By continuity of ρ_q (Corollary 1.19), $\lim_i \epsilon_{\mathcal{X}}(t_i)q = pq = p$.

Now it follows from $1_T \in \mathcal{X}$ that $\operatorname{cl}_{G^{\mathcal{X}}} T$ is open, hence there must be some $t \in T$ with $\epsilon_{\mathcal{X}}(t)q = v \in \operatorname{cl}_{G^{\mathcal{WAP}}} T$. Since $b^{\mathcal{X}}(q) = 1_G$ ($b^{\mathcal{X}}(pq) = b^{\mathcal{X}}(p)$ and $G^{\mathcal{AP}}$ is a group), we have that $b(\epsilon_{\mathcal{X}}(t)) = b^{\mathcal{X}}(v)$, with $v \in \operatorname{cl}_{G^{\mathcal{X}}} T$. By Lemma 4.9 $v = \epsilon_{\mathcal{X}}(t) \in \epsilon_{\mathcal{X}}(G)$. Since b is injective we finally conclude that $t = g$. From $\epsilon_{\mathcal{X}}(t)q = v = \epsilon_{\mathcal{X}}(t)$, we conclude that $q = 1_G$ as desired. $\square$

COROLLARY 4.11. *Let G denote a locally compact Abelian group. If $T \subset G$ is an I_0-set and $p \in \operatorname{cl}_{G^{\mathcal{X}}} T$, then $px \neq p$ for every $1_G \neq x \in G^{\mathcal{X}}$. where $\mathcal{X}$ is any of the algebras $\mathcal{LUC}(G)$, $\mathcal{WAP}(G)$ and $\mathcal{B}(G)$.*

Proof. This corollary follows immediately from Theorem 4.10 and the fact that Sidon sets (and therefore I_0-sets) are uniformly approximable Sidon sets, Statement (2) of Theorem 3.7. $\square$

While the algebraic structure of $G^{\mathcal{LUC}}$ is more involved, its topological structure is simpler, and this helps in unveiling some algebraic relations, let us see how a well-known Lemma on the topology of βG leads to finding cancellable elements in $G^{\mathcal{LUC}}$.

Lemma 4.12 (Theorem 3.40 of [**HS98**]). *Let A and B are countable (or σ-compact) subsets of βD. If $\mathrm{cl}_{\beta D} A \cap B = A \cap \mathrm{cl}_{\beta D} B = \emptyset$, then $\mathrm{cl}_{\beta D} A \cap \mathrm{cl}_{\beta D} B = \emptyset$.*

Corollary 4.13. *Let G be a discrete maximally almost periodic group with uncountably many finite dimensional inequivalent representations. Then $G^{\mathcal{LUC}} = \beta G$ has left cancellable elements, that is there are elements $p \in \beta G$ such that*

$$q \neq r \in G^{\mathcal{LUC}} \implies pq \neq pr.$$

Proof. By theorem 3.20 G must contain some countably infinite I_0-set A. Take $p \in \mathrm{cl}_{\beta G} A$ and let $q, r \in \beta G$ be such that $pq = pr$. If $Aq \cap \mathrm{cl}_{\beta D} Ar = \emptyset$ and $\mathrm{cl}_{\beta D} Aq \cap Ar = \emptyset$, then $\mathrm{cl}_{G^{\mathcal{LUC}}} Aq \cap \mathrm{cl}_{G^{\mathcal{LUC}}} Ar = \emptyset$, by Lemma 4.12. Since $pq \in \mathrm{cl}_{G^{\mathcal{LUC}}} Aq \cap \mathrm{cl}_{G^{\mathcal{LUC}}} Ar$, we can assume that $Aq \cap \mathrm{cl}_{G^{\mathcal{LUC}}} Ar \neq \emptyset$. This means that there are $a \in A$ and $v \in \mathrm{cl}_{G^{\mathcal{LUC}}} A$, $aq = vr$.

We have as in Theorem 4.10 that $b^{\mathcal{LUC}}(q) = b^{\mathcal{LUC}}(r)$ and thus that $b(a) = b^{\mathcal{LUC}}(r)$. Lemma 4.9 applies and shows that $r = \epsilon_{\mathcal{LUC}}(a)$. Since a and r were chosen so that $aq = vr$ and $a \in G$ is invertible we conclude that $q = r$. $\square$

Theorems 4.10 and 4.13, and Corollary 4.11 are not sharp. Much more on cancellability on $G^{\mathcal{LUC}}$ is known. As we have said $G^{\mathcal{LUC}}$ is not commutative and as a consequence left and right cancellability are different concepts. We have here chosen to illustrate left cancellability because its techniques adapt better to other compactifications. But, according [**HS98**, p. 174] right cancellability is easier to deal with than left cancellability due to continuity of right translations ρ_q. We summarize here, without proof what Hindman and Strauss, [loc. cit.], prove about right cancellability, we refer the reader to this monograph for unexplained terminology.

Theorem 4.14 (Sections 8,9 and 10 of [**HS98**]). *Let D be a discrete group.*

(1) *If $xp \neq p$ for every $1_D \neq x \in \beta D$, then p is right cancellable.*

(2) *If D is countable, $x \in \beta D$ is right cancellable if and only if Dx is discrete.*
(3) *The set of right cancellable elements of βD contains an open and dense subset in βD.*
(4) *Weak P-points in βD are right cancellable. P-points are both left and right cancellable.*
(5) *There are left cancellable elements that are not right cancellable.*

Cancellability on $G^{\mathcal{LUC}}$ for G not discrete has also been studied, see Filali [**Fil96**]. One intriguing question that seems to be open is:

Question 9 (Question 8.43 of [**HS98**]). Are weak P-points left cancellable in $\beta\mathbb{N}$?

Interpolation sets have proved essential to find points in $G^{\mathcal{WAP}}$ or $G^{\mathcal{B}}$ such that $px \neq p$ for every other point x in the compactification. Idempotents constitute the easiest example of noncancellable elements. Idempotents occupy a region of the compactification inaccessible for interpolation sets: if interpolation sets can be regarded as *thin* sets, idempotents are in the closure of rather *thick* sets:

Theorem 4.15. *Let $p \in \mathbb{Z}^{\mathcal{LUC}}$ be an idempotent. If $p \in \mathrm{cl}_{\mathbb{Z}^{\mathcal{LUC}}} A$ with $A \subset \mathbb{Z}$, then $A \cap n\mathbb{Z} \neq \emptyset$ for every $n \in \mathbb{Z}$.*

Proof. Consider the quotient homomorphism $Q_n\colon \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. Since $\mathbb{Z}/n\mathbb{Z}$ is finite, Q extends to a continuous homomorphism $\bar{Q_n}\colon \mathbb{Z}^{\mathcal{LUC}} \to \mathbb{Z}_n$. We have then that $\bar{Q_n}(p) = 0$. Thus $\bar{Q_n}^{-1}(\{0\})$ is a neighbourhood of p and $A \cap \bar{Q_n}^{-1}(\{0\}) = A \cap n\mathbb{Z} \neq \emptyset$. □

This however does not seem sufficient to prove cancellability,

Question 10 (Filali). Are there any cancellable elements in $\mathbb{Z}^{\mathcal{WAP}}$ or $\mathbb{Z}^{\mathcal{B}}$?

We close this Section remarking that the same circle of ideas taking to Theorem 4.10 and its consequences leads to a considerable extension of Veech's theorem off G. The hypothesis of the following theorem are not sharp, this is just a sample.

Theorem 4.16. *Let G be a maximally almost periodic group with uncountably many inequivalent finite dimensional representations. For every C^*-algebra with $\mathcal{LUC}(G) \supset \mathcal{X} \supset \mathcal{B}(G)$, there is an open and dense subset V of $G^{\mathcal{X}}$ such that for every $p \in G^{\mathcal{X}}$, the mapping ρ_p is injective on U.*

Sketch. Let $A \subset G$ be an I_0-set. Then $U := \mathrm{cl}_{\mathcal{X}} A \cup G$ will be open and dense in $G^{\mathcal{X}}$ (note that $1_A \in \mathcal{B}(G) \subset \mathcal{X}$). If $u, v \in \mathrm{cl}_{\mathcal{X}} A$, and $up = vp$ we have as usual that $b^{\mathcal{X}}(u) = b^{\mathcal{X}}(v)$. An argument similar to that of Lemma 4.9 then shows that $u = v$. Veech's theorem takes care of the remaining cases and the theorem follows. □

In [**BP06**] Budak and Pym obtain the analog of Theorem 4.16 for $G^{\mathcal{LUC}}$, we see here how its main hypothesis ($b\colon G \to G^{\mathcal{AP}}$ is not surjective) relates to our previous arguments.

Corollary 4.17 (Theorem 4.1 of [**BP06**]). *If G is a locally compact σ-compact group for which $b\colon G \to G^{\mathcal{AP}}$ is not surjective, then $G^{\mathcal{LUC}} = \beta G$ contains an open and dense subset U such that for every $p \in G^{\mathcal{LUC}}$, the mapping ρ_p ($\rho_p(u) = up$) is injective on U.*

Proof of the discrete case (sketch). The proof is much the same as that of Theorem 4.16. Since b is not surjective and $b(G)$ is σ-compact, $b(G)$ cannot be countably compact and a sequence $B = (x_n)$ can be found such that $b(B)$ is discrete.

Suppose now that $u_1 p = u_2 p$ with $u_1 \neq u_2 \in B$. Take $B_j \subset B$, $j = 1, 2$, with $u_j p \in \rho_p(\mathrm{cl}_{G^{\mathcal{LUC}}} B_j)$ and $B_1 \cap B_2 = \emptyset$. We can assume, by Lemma 4.12, that there is some $g_0 p \in B_1 p \cap (\mathrm{cl}_{G^{\mathcal{LUC}}} B_2) p \neq \emptyset$. Then $b(g_0) b^{\mathcal{LUC}}(p) = b^{\mathcal{LUC}}(g_0 p) = b^{\mathcal{LUC}}(qp)$ for some $q \in \mathrm{cl}_{G^{\mathcal{LUC}}} B_2$, since $G^{\mathcal{AP}}$ is a group we conclude that $b(g_0) = b^{\mathcal{LUC}}(q)$ and Lemma 4.9 yields $g_0 = q$, a contradiction with $B_1 \cap B_2 \neq \emptyset$. □

Remark 4.18. In theorems 4.13 and 4.16 we have made use of I_0-sets. The present proofs do not allow to use just Sidon or $\mathcal{WAP}$-interpolation sets (as could be guessed). Let A_1 and A_2 be disjoint I_0-sets such that $\mathrm{cl}_{G^{\mathcal{AP}}} A_1 \cap \mathrm{cl}_{G^{\mathcal{AP}}} A_2 \neq \emptyset$ but $A_0 = A_1 \cup A_2$ is not I_0, cf. Example 3.5. Let $p \in G^{\mathcal{WAP}}$ (or $G^{\mathcal{B}}$) be such that $b^{\mathcal{WAP}}(p) \in \mathrm{cl}_{G^{\mathcal{AP}}} A_1 \cap \mathrm{cl}_{G^{\mathcal{AP}}} A_2$. This will produce elements p_1, p_2 in the $G^{\mathcal{WAP}}$-closure of A_1 and A_2 (necessarily different as A_0 *is* Sidon) with $b^{\mathcal{WAP}}(p_1) = b^{\mathcal{WAP}}(p_2)$, so that the argument of Theorem 4.16 does not work. In theorem 4.13 we used that I_0-sets are discrete in $G^{\mathcal{AP}}$. It is an old open question whether Sidon sets of $\mathbb{Z}$ can accumulate at some point of $\mathbb{Z}$.

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