REFLEXIVITY OF PRODISCRETE TOPOLOGICAL GROUPS

J. GALINDO(1), L. RECODER-NÚÑEZ(2), AND M. TKACHENKO(3)

Abstract. We study the duality properties of two rather different classes of subgroups of direct products of discrete groups (prodiscrete groups): P-groups, i.e., topological groups such that countable intersections of its open subsets are open, and protodiscrete groups of countable pseudocharacter (topological groups in which the identity is the intersection of countably many open sets).

It was recently shown by the same authors that the direct product \( \Pi \) of an arbitrary family of discrete Abelian groups becomes reflexive when endowed with the \( \omega \)-box topology. This was the first example of a non-discrete reflexive P-group. Here we present a considerable generalization of this theorem and show that every product of feathered (equivalently, almost metrizable) Abelian groups equipped with the P-modified topology is reflexive. In particular, every locally compact Abelian group with the P-modified topology is reflexive. We also examine the reflexivity of dense subgroups of products \( \Pi \) with the P-modified topology and obtain the first examples of non-complete reflexive P-groups.

We find as well that the better behaved class of prodiscrete groups (complete protodiscrete groups) of countable pseudocharacter contains non-reflexive members—any uncountable bounded torsion Abelian group \( G \) of cardinality \( 2^\omega \) supports a topology \( \tau \) such that \((G,\tau)\) is a non-reflexive prodiscrete group of countable pseudocharacter.

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1. Introduction

Following the terminology of [16] we say that a topological group is protodiscrete provided its identity has a neighbourhood basis of open subgroups. The term linear groups can also be found in the literature. Projective limits of discrete groups correspond then to complete protodiscrete groups. These groups are usually called prodiscrete.

The duality theory of prodiscrete groups seems to be far from complete as of today. While the case of projective limits of (countable) sequences of groups is relatively easy to deal with, for closed subgroups of countable products of discrete groups are always reflexive (they are metrizable and, e.g., [4, Theorem 17.3] applies), the case of uncountable inverse limits still appears rather mysterious. For years, the only uncountable projective limit of discrete groups with a known dual behavior (not covered by Kaplan’s product theorem in [17]) was the direct sum, also called \(\sigma\)-product, of \(\omega_1\) copies of the two-element group \(\mathbb{Z}_2\) equipped with the \(\omega\)-box topology. Leptin proved in [18] that the second dual of this group is discrete, providing thus the first example of a non-reflexive projective limit of discrete groups. This topological group has the additional property of being a \(P\)-space: countable intersections of its open subsets remain open. Topological groups whose underlying topological space is a \(P\)-space are usually called \(P\)-groups and, as a matter of fact, every Abelian \(P\)-group is protodiscrete.

At the opposite end of \(P\)-groups we find the groups of countable pseudocharacter (that is, those topological groups whose identity element is the intersection of countably many open sets). Only discrete groups are to be found in both classes. Reflexive prodiscrete groups of countable pseudocharacter are easy to come by, as every metrizable prodiscrete group is reflexive. \(P\)-groups on the other hand have been regarded as being more prone to nonreflexivity, basically due to Leptin’s example quoted above. However non-discrete reflexive \(P\)-groups do exist, the first examples were recently obtained in [12].

In this paper we give a closer look to both classes of protodiscrete groups. We find more examples of reflexive \(P\)-groups and the first examples (to the best of our knowledge) of non-reflexive prodiscrete groups of countable pseudocharacter.
In Section 3 we extend considerably the collection of known reflexive $P$-groups. We show that
the class of groups whose $P$-modification is reflexive contains all feathered groups and arbitrarily
large products of feathered groups. In particular, the $P$-modification of every locally compact
Abelian group is reflexive. This fact is new even for compact groups and it can be considered as a
natural complement to the Pontryagin–van Kampen duality theorem. However, the $P$-modification
of a reflexive group $G$ can fail to be reflexive, even if $G$ is $\sigma$-compact [6].

With the aim of outlining the borders between reflexivity and non-reflexivity in $P$-groups we
find, in Proposition 4.14 and Theorem 4.16, a series of reflexive proper dense subgroups of the
$P$-modification of the compact group $\mathbb{Z}_\tau^+$, with $\tau > \omega$. These are the first examples of reflexive $P$-
groups that fail to be complete (all reflexive $P$-groups obtained in [12] were complete). A “simpler”
series of non-complete reflexive $P$-groups is presented in Corollary 4.12. However, the proof of the
reflexivity of these groups is based on a reflection principle (Theorem 4.9) that reduces the problem
of the reflexivity of a big subgroup of a product group to the study of the reflexivity of projections
of the subgroup to relatively small subproducts.

It should also be mentioned that the class of reflexive $P$-groups has unexpectedly good perma-
nence properties. For example, it was shown in [12] that this class is closed with respect to taking
quotients. Another property is presented here in Proposition 4.10. It turns out that if a $P$-group
$H$ contains a dense reflexive subgroup, then $H$ itself is reflexive. Simple examples show that these
facts are not valid outside the class of protodiscrete groups.

Section 5 addresses the class of prodiscrete groups of countable pseudocharacter. We show
that every bounded torsion Abelian group of cardinality $2^\omega$ admits a nondiscrete Hausdorff group
topology $\tau$ of countable pseudocharacter such that $(G, \tau)$ is protodiscrete and non-reflexive.

Finally, in Section 6 we show that if $G$ is a dense subgroup of a $P$-group $H$, then the canonical
continuous isomorphism of $H^\wedge$ onto $G^\wedge$ obtained by restricting to $H$ of the characters of $G$ is a
homeomorphism if and only if $G = H$. In other words, no proper dense subgroup of a $P$-group
determines it.

We finish the article with several open problems, collected in Section 7, the solution to which
can improve substantially our understanding of Pontryagin’s duality in the class of $P$-groups.
2. Notation

All groups considered here are assumed to be Abelian if otherwise is not specified explicitly. The complex plane with its usual multiplication and topology is denoted by $\mathbb{C}$. A \textit{character} on a group $G$ is a homomorphism of $G$ to the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The group of all \textit{continuous} characters of $G$ equipped with the pointwise multiplication and topology of uniform convergence on the compact subsets of $G$ will be called the \textit{dual group} of $G$ and denoted by $G^\wedge$. Then $G^\wedge$ is again a topological Abelian group, and its dual group $(G^\wedge)^\wedge$ will be referred to as the \textit{bidual group} of $G$ and denoted by $G^{\wedge\wedge}$.

The \textit{evaluation homomorphism} $\alpha_G : G \to G^{\wedge\wedge}$ is defined by $\alpha_G(x)(\chi) = \chi(x)$, for all $x \in G$ and $\chi \in G^\wedge$. The group $G$ is called \textit{reflexive} if the homomorphism $\alpha_G : G \to G^{\wedge\wedge}$ is a topological isomorphism.

Given a topological group $H$, we denote by $PH$ the \textit{$P$-modification} of $H$ which is the same underlying group $H$ endowed with the finer topological group topology whose base is formed by the family of $G_\delta$-sets in the original group $H$. Some basic information about $P$-groups can be found in [2, Section 4.4].

The subgroup of a group $H$ generated by a set $A \subset H$ is $\langle A \rangle$. The neutral element of $H$ will be denoted by $0_H$, except for the circle group $\mathbb{T}$ whose identity is $1$.

We say that a group $G$ is \textit{bounded torsion} if there is a positive integer $n$ such that $ng = 0_G$ for all $g \in G$.

A topological group $G$ is \textit{feathered} if $G$ contains a nonempty compact subset with a countable neighbourhood base in $G$. It is known that $G$ is feathered iff $G$ contains a compact subgroup $K$ such that the quotient space $G/K$ is metrizable [20]. We also say that $G$ is \textit{$\omega$-bounded} if the closures of countable subsets of $G$ are compact.

Let $\{D_i : i \in I\}$ be a family of topological groups and $D = \prod_{i \in I} D_i$ the product group with the usual Tychonoff product topology. Elements of $D$ will be regarded as functions $x : I \to \bigcup_{i \in I} D_i$ such that $x(i) \in D_i$ for all $i \in I$. If $x \in D$, we define the \textit{support} of $x$ as

$$\text{supp}(x) = \{i \in I : x(i) \neq 0_i\},$$

where $0_i$ is the neutral element of $D_i$. With these notations, the subgroup

$$\Sigma D = \{x \in D : |\text{supp}(x)| \leq \omega\}$$
of $D$ is called the $\Sigma$-product of the family $\{D_i : i \in I\}$. Similarly,

$$\sigma D = \{x \in D : |\text{supp}(x)| < \omega\}$$

is a subgroup of $D$ which is called the $\sigma$-product of the family $\{D_i : i \in I\}$. Clearly, $\Sigma D$ is a dense subgroup of $PD$, while $\sigma D$ is a dense subgroup of $D$.

Let $[I]^{\leq \omega}$ denote the family of all countable subsets of the index set $I$. In what follows we will use the sets

$$U(J) = \{x \in D : x(i) = 0, \text{ for all } i \in J\},$$

with $J \subset I$. Notice that if the groups $D_i$'s are discrete, then the family $\{U(J) : J \in [I]^{\leq \omega}\}$ forms a local base at the identity of $PD$ and each $U(J)$ is a subgroup of $PD$.

We will mainly be working with topological groups $G$ such that

$$\Sigma D \subseteq G \subseteq PD.$$ 

The subgroup $\Sigma D$ of $PD$ will always carry the topology inherited from $PD$, i.e., $\Sigma D$ is a $P$-group.

Let $G$ be a subgroup of $PD$ and $J \subset I$. We will say that a character $\chi : G \to \mathbb{T}$ depends on $J$ if there is $x \in G$ with $\text{supp}(x) \subset J$ such that $\chi(x) \neq 1$; notice that $x \in G \cap U(I \setminus J)$.

It is easy to verify (see [12, Lemma 4.1]) that for every character $\chi \in G^\wedge$, one can find $J \in [I]^{\leq \omega}$ and a continuous character $\chi_J$ on the subgroup $\pi_J(G)$ of $PD_J$ such that $\chi = \chi_J \circ \pi_J$. Here $\pi_J$ is the projection of $D$ to $D_J = \prod_{i \in J} D_i$. Therefore, for every character $\chi \in G^\wedge$, there is $J \in [I]^{\leq \omega}$ such that $\chi$ does not depend on $I \setminus J$. We say in this case that $\chi$ depends on countably many coordinates. Similarly, we say that a set $K \subset G^\wedge$ depends on countably many coordinates if there is $J \in [I]^{\leq \omega}$ such that every $\chi \in K$ does not depend on $I \setminus J$.

3. Background

Prodiscrete groups, being subgroups of products of discrete groups, are examples of nuclear groups [4] and, hence, they have a rich duality theory. The combination of Theorems 8.5, 14.3, and 17.3 of [4] and Corollary 6.5 of [10] (or Corollary 21.5 of [3]) implies the following result.

**Theorem 3.1.** Among protodiscrete groups, the evaluation homomorphism $\alpha_G : G \to G^{\wedge\wedge}$ has the following properties:

1. $\alpha_G$ is a relatively open injective homomorphism, i.e., $\alpha_G$ is open as a mapping onto $\alpha_G(G)$.
2. If $G$ is prodiscrete, then $\alpha_G$ is an open isomorphism.
(3) If $G$ is metrizable and prodiscrete, then $\alpha_G$ is a topological isomorphism, i.e., $G$ is reflexive.

In the rest of this section we supply the reader with necessary information about reflexivity in $P$-groups taken mainly from [12]. We begin with the following characterization of reflexivity for $P$-groups and its specialization for $P$-modifications of products.

**Theorem 3.2** (Theorem 3.2 and Corollary 3.3 of [12]). Let $G$ be a $P$-group. The evaluation mapping $\alpha_G : G \to G^{\wedge\wedge}$ is always an open isomorphism. The group $G$ is reflexive if and only if every compact set $K \subset G^{\wedge}$ is constant on an open subgroup of $G$.

It is immediate from Theorem 3.2 that a $P$-group $G$ is reflexive if and only if the evaluation homomorphism $\alpha_G$ is continuous. If $G$ is a subgroup of a product group with the $P$-modified topology, the above theorem acquires the form given below:

**Corollary 3.3** (Theorem 3.4 of [12]). Let $D = \prod_{i \in I} D_i$ be a product of topological groups. A subgroup $G$ of $PD$ is reflexive if and only if for every compact set $K \subset G^{\wedge}$, there exist a set $J \in [I]^{\leq \omega}$ and an open subgroup $U$ of $PD_J$ such that $K$ is constant on $G \cap \pi_J^{-1}(U)$.

The following result shows that the reflexivity of uncountable products of topological groups with the $P$-modified topology is completely determined by countable subproducts; it can be called as the first reflection principle:

**Theorem 3.4** (Proposition 4.7 of [12]). Let $D = \prod_{i \in I} D_i$ be a product of topological groups. Then the group $PD$ is reflexive if and only if $PD_J$ is reflexive for each $J \in [I]^{\leq \omega}$.

Another reflection principle, designed for special dense subgroups of $PD$, will be presented in Theorem 4.9.

We finish this section with a theorem about the existence of non-discrete reflexive $P$-groups which also serves for us as the point of departure:

**Theorem 3.5** (Theorem 4.8 of [12]). Let $D = \prod_{i \in I} D_i$ be a product of discrete Abelian groups. Then the $P$-group $PD$ and the $\omega$-bounded group $(PD)^{\wedge}$ are reflexive. If the product $D$ contains uncountably many nontrivial factors, then $PD$ is non-discrete while the dual group $(PD)^{\wedge}$ is non-compact.
4. Products of feathered groups and their $P$-modifications

We find in this section a considerably wider class of reflexive $P$-groups than $P$-modifications of products of discrete groups (see Theorem 3.5 above). Our technique is summarized in the following two lemmas proved in [12].

Throughout this section, $D$ stands for the product $\prod_{i \in I} D_i$ of an arbitrary family of topological Abelian groups with the usual Tychonoff product topology, while $G$ denotes a (topological) subgroup of $PD$, the $P$-modification of $D$. The set $\{e^{\pi ix} : -1/2 < x < 1/2\}$ will be denoted by $\mathbb{T}_+$. 

**Lemma 4.1** (Proposition 4.3 of [12]). Suppose that $C = \{\chi_\eta : \eta < \omega_1\} \subset (PD)^\wedge$, $\mathcal{J} = \{J_\eta : \eta < \omega_1\} \subset [I]^\omega$, and $X = \{x_\eta : \eta < \omega_1\} \subset PD$ satisfy the following conditions for each $\eta < \omega_1$:

1. $\chi_\eta$ does not depend on $I \setminus J_\eta$;
2. $\text{supp}(x_\eta) \subset J_\eta$;
3. $\chi_\eta(x_\eta) \in T \setminus T_+$;
4. if $\zeta < \eta$, then $J_\zeta \cap \text{supp}(x_\eta) = \emptyset$.


Then every element of $\bigcap_{\gamma < \omega_1} \overline{\{\chi_\eta : \eta \geq \gamma\}}^{PD}$ is discontinuous as a character on $PD$.

**Lemma 4.2** (Lemma 4.5 of [12]). Suppose that $\Sigma D \subseteq G \subseteq PD$ and $K \subseteq G^\wedge$. If $K$ depends on uncountably many coordinates, then $K$ contains a subset $C = \{\chi_\eta : \eta < \omega_1\}$ and $G$ contains a subset $X = \{x_\eta : \eta < \omega_1\}$ such that conditions (1)–(4) of Lemma 4.1 hold for $C$, $X$, and a suitable collection $\mathcal{J} = \{J_\eta : \eta < \omega_1\}$ of countable subsets of the index set $I$.

We need the following simple fact to establish the reflexivity of $P$-modifications of compact Abelian groups in Proposition 4.4:

**Lemma 4.3.** Suppose that $\pi : G \to H$ is a continuous onto homomorphism of compact groups, not necessarily Abelian. Then the homomorphism $\pi : PG \to PH$ is open, where $PG$ and $PH$ are the $P$-modifications of the groups $G$ and $H$, respectively.

**Proof.** Let $e$ be the neutral element of $G$. It is clear that the sets of the form $V = \bigcap_{n \in \omega} U_n$, where $U_n$’s are open neighbourhoods of $e$ in $G$ and $\overline{U}_{n+1} \subset U_n$ for each $n \in \omega$ (the closure is taken in $G$), constitute a base at $e$ in $PG$. Therefore, it suffices to verify that every image $\pi(V)$ is open in $PH$. Notice that the continuous epimorphism $\pi : G \to H$ is open since $G$ is compact. Using the
compactness of $G$ once again we see that $\pi(V) = \bigcap_{n \in \omega} \pi(U_n)$, so $\pi(V)$ is a $G_\delta$-set in $H$. Hence $\pi(V)$ is open in $PH$. \hfill \Box

**Proposition 4.4.** The $P$-modification $PH$ of every compact Abelian group $H$ is reflexive.

**Proof.** It is well known that one can find a compact Abelian group $G$ of the form $G = \prod_{i \in I} G_i$, with compact metrizable factors $G_i$, and a continuous homomorphism $\pi$ of $G$ onto $H$ (see [15, Lemma 1.6]). By Lemma 4.3, the homomorphism $\pi : PG \to PH$ is open. For every $i \in I$, let $D_i$ be the group $G_i$ with the discrete topology. Denote by $D$ the product group $\prod_{i \in I} D_i$. Since the factors $G_i$ are metrizable, the topological groups $PG$ and $PD$ coincide. It now follows from Theorem 3.5 that the group $PG$ is reflexive. According to [12, Proposition 3.5], open continuous homomorphisms preserve reflexivity in the class of $P$-groups, which implies the reflexivity of $PH$. \hfill \Box

It turns out that Proposition 4.4 cannot be extended to $\sigma$-compact groups. According to [6], the free Abelian topological group $A(X)$ over the one-point compactification $X$ of an uncountable discrete space is reflexive and $\sigma$-compact, but the $P$-modification of $A(X)$ is not reflexive. Moreover, the second dual of $PA(X)$ is discrete.

Let us call a topological group $H$ *pseudo-feathered* if there exists a nonempty compact set of type $G_\delta$ in $H$. It is clear that every feathered group is pseudo-feathered and that $H$ is pseudo-feathered if and only if it contains a compact subgroup of type $G_\delta$ (if $G$ contains a $G_\delta$-set $K = \bigcap_n V_n$, with $V_n$ open, and $k \in K$, then we may choose open neighbourhoods $W_n$ of the identity with $W_{n+1}^2 \subseteq W_n \cap (-k + V_n)$, then $\bigcap_n W_n$ is the desired compact subgroup, cf. [2, Lemma 4.3.10]). An Abelian group is pseudo-feathered if it admits an open continuous homomorphism with compact kernel onto a group whose identity is a $G_\delta$-set. In the following result we extend the conclusion of Proposition 4.4 to pseudo-feathered groups.

**Proposition 4.5.** Let $H$ be a pseudo-feathered Abelian group. Then the group $PH$ is reflexive.

In particular, the $P$-modification of every locally compact group is reflexive.

**Proof.** Let $C$ be a compact subgroup of type $G_\delta$ in $H$. Clearly, $PC$ is an open subgroup of $PH$. Since a topological group containing an open reflexive subgroup is itself reflexive (see Proposition 2.2 of [5]) and the group $PC$ is reflexive by Proposition 4.4, we conclude that $PH$ is reflexive as well. \hfill \Box

The next result is a common generalization of Theorem 3.5 and Proposition 4.5:
Theorem 4.6. Let $H = \prod_{i \in I} H_i$ be the product of a family of pseudo-feathered Abelian groups. Then the group $PH$ is reflexive.

Proof. It is easy to verify that if $C_n$ is a compact set of type $G_\delta$ in a space $X_n$, for each $n \in \omega$, then the compact set $C = \prod_{n \in \omega} C_n$ has type $G_\delta$ in the product space $\prod_{n \in \omega} X_n$. This observation implies that the group $H_J = \prod_{i \in J} H_i$ is pseudo-feathered for each $J \in [I]^{\omega}$. The reflexivity of $PH$ now follows from Proposition 4.5 and Theorem 3.4. □

In Theorem 4.9 below we characterize the reflexivity of certain subgroups $G$ of “big” products $PD = \prod_{i \in I} D_i$ of topological groups in terms of projections $\pi_J(G)$ of $G$ to relatively small subproducts $PD_J = \prod_{i \in J} D_i$. First we need two lemmas.

Lemma 4.7. If $\Sigma D \subset G \subset PD$, then the restriction to $G$ of the projection $\pi_J \colon PD \to PD_J$ is an open homomorphism of $G$ onto $\pi_J(G)$, for every nonempty set $J \subset I$.

Proof. Let $J$ be a nonempty subset of $I$. Since $G$ is a subgroup of $PD$, the restriction of $\pi_J$ to $G$ is a continuous homomorphism. Hence it suffices to verify that the image $\pi_J(V \cap G)$ is open in $\pi_J(G)$, for every basic open neighbourhood $V$ of the identity $e$ in $PD$. In fact, we will show that $\pi_J(V \cap G) = \pi_J(V) \cap \pi_J(G)$.

Given a basic open neighbourhood $V$ of $e$ in $PD$, one can find a countable set $C \subset I$ and $G_\delta$-sets $V_i$ in $D_i$ for $i \in C$ such that

$$V = \{ x \in D : x(i) \in V_i \text{ for each } i \in C \}.$$ 

Let $F = C \cap J$ and $E = C \setminus J$. It is clear that $F$ and $E$ are disjoint countable sets and $C = F \cup E$. Take an arbitrary point $y \in \pi_J(V) \cap \pi_J(G)$. There exists an element $x \in G$ with $\pi_J(x) = y$. Clearly, $x(i) = y(i) \in V_i$ for each $i \in F$. Since $E$ is countable, we can find an element $x_0 \in \Sigma D$ such that supp$(x_0) \cap J = \emptyset$ and $x_0(i) = x(i)$ for each $i \in E$. Then the element $z = x \cdot x_0^{-1}$ of $D$ satisfies $z(i) \in V_i$ for each $i \in C$, so $z \in V$. Since $x_0 \in \Sigma D \subset G$, we see that $z \in G$. Hence $z \in V \cap G$ and $\pi_J(z) = \pi_J(x) \cdot (\pi_J(x_0))^{-1} = y$. This implies that $\pi_J(V \cap \pi_J(G)) \subset \pi_J(V \cap G)$. The inverse inclusion is obvious, so the equality $\pi_J(V \cap G) = \pi_J(V) \cap \pi_J(G)$ is proved. Therefore the restriction of the homomorphism $\pi_J$ to $G$ is open when considered as a mapping onto its image. □

In the following lemma we collect well-known properties of dual homomorphisms that will be used in the proof of Theorem 4.9.
Lemma 4.8. Let $\pi: G \to H$ be a continuous homomorphism of topological Abelian groups. Let also $\pi^\wedge: H^\wedge \to G^\wedge$ be the dual homomorphism defined by $\pi^\wedge(\chi) = \chi \circ \pi$, for each $\chi \in H^\wedge$. Then:

(a) $\pi^\wedge$ is continuous.
(b) If $\pi$ is compact covering, then $\pi^\wedge$ is a topological isomorphism of $H^\wedge$ onto its image $\pi^\wedge(H^\wedge)$.
(c) If the homomorphism $\pi$ is open, then the image $\pi^\wedge(H^\wedge)$ is closed in $G^\wedge$.

Proof. Only (c) needs to be proved, as (a) and (b) are well known (see for instance [3, Lemma 5.17]). Suppose that $\pi$ is an open homomorphism and let $K = \pi(G)$. Then $K$ is an open subgroup of $H$, so every continuous character on $K$ extends to a continuous character on $H$. Hence $\pi^\wedge(K^\wedge) = \pi^\wedge(H^\wedge)$ and we can assume without loss of generality that $\pi(G) = H$. Finally, since $\pi$ is open, the image $\pi^\wedge(H^\wedge)$ coincides with the annihilator of the kernel of $\pi$ in $G^\wedge$, and the latter subgroup is always closed. \[\square\]

Here is the second reflection principle (after Theorem 3.4) which serves for establishing the reflexivity of certain dense subgroups $G$ of large products $P \prod_{i \in I} D_i$. It reduces the problem of whether $G$ is reflexive to the study of reflexivity of the projections $\pi_J(G) \subset PD_J$, with $|J| \leq \aleph_1$.

Notice that we do not require the equality $\pi_J(G) = D_J$ for subsets $J$ of $I$ satisfying $|J| \leq \aleph_1$.

Theorem 4.9. Suppose that $D = \prod_{i \in I} D_i$ is a product of topological groups and $\Sigma D \subset G \subset PD$. Then the group $G$ is reflexive iff the subgroup $\pi_J(G)$ of $PD_J$ is reflexive, for each set $J \subset I$ satisfying $|J| \leq \aleph_1$.

Proof. Necessity. Let $G$ be reflexive. Take any $J \subset I$ satisfying $|J| \leq \aleph_1$ and put $H = \pi_J(G)$. By Lemma 4.7, the restriction to $G$ of the projection $\pi_J: PD \to PD_J$ is an open homomorphism of $G$ onto $H$. Hence the reflexivity of $H$ follows from [12, Proposition 3.5].

Sufficiency. Suppose that $\pi_J(G)$ is reflexive, for each $J \subset I$ with $|J| \leq \aleph_1$. Since $\Sigma D \subset G$, the equality $\pi_J(G) = D_J$ holds for each $J \in [I]^{\leq \omega}$. Therefore, given a compact set $K \subset G^\wedge$, it suffices to find a countable set $J \subset I$ and an open subgroup $U$ of $PD_J$ such that $K$ is constant on $G \cap \pi_J^{-1}(U)$ (see Corollary 3.3). First, we claim that $K$ depends at most on countably many coordinates. If not, apply Lemma 4.2 to choose families $\{\chi_\eta: \eta < \omega_1\} \subset K$, $\{x_\eta: \eta < \omega_1\} \subset G$, and $\{J_\eta: \eta < \omega_1\} \subset [I]^{\leq \omega}$ satisfying conditions (1)--(4) of Lemma 4.1. Let $J = \bigcup_{\eta < \omega_1} J_\eta$. Then $J \subset I$ and $|J| \leq \aleph_1$. Hence the
subgroup $H = \pi_J(G)$ of $PD_J$ is reflexive. Notice that by Lemma 4.7, the restriction to $G$ of the homomorphism $\pi_J$ is open when considered as a mapping of $G$ onto $H$. Let $\eta < \omega_1$. Since $\chi_\eta$ does not depend on $I \setminus J_\eta$ and $J_\eta \subseteq J$, there exists a continuous character $\psi_\eta$ on $H$ such that $\chi_\eta = \psi_\eta \circ \pi_J|_G$. We put $\Psi = \{\psi_\eta : \eta < \omega_1\}$.

Denote by $\varphi$ the continuous homomorphism $(\pi_J|_G)^\wedge$ of $H^\wedge$ to $G^\wedge$. Then $\varphi(\psi_\eta) = \chi_\eta \in K$ for each $\eta < \omega_1$, so $\varphi(\Psi) \subseteq K$. Since $H$ is a $P$-group, all compact subsets of $H$ are finite. It now follows from (b) of Lemma 4.8 that $\varphi$ is a topological isomorphism of $H^\wedge$ onto the subgroup $\varphi(H^\wedge)$ of $G^\wedge$. Further, since the homomorphism $\pi_J|_G$ of $G$ onto $H$ is open, (c) of Lemma 4.8 implies that $\varphi(H^\wedge)$ is a closed subgroup of $G^\wedge$. Therefore, $C = K \cap \varphi(H^\wedge)$ is a compact subset of $\varphi(H^\wedge)$ and $L = \varphi^{-1}(C)$ is a compact subset of $H^\wedge$. It follows from $\Psi \subseteq H^\wedge$ and $\varphi(\Psi) \subseteq K$ that $\Psi \subseteq L$.

The latter inclusion and the definition of the set $\Psi$ together imply that $L$ depends on uncountably many coordinates.

Indeed, suppose that for some countable set $A \subseteq J$, every element of $L$ does not depend on $J \setminus A$. In particular, $\psi_\eta$ does not depend on $J \setminus A$, for each $\eta < \omega_1$. Since $\chi_\eta = \psi_\eta \circ \pi_J|_G$, we see that each $\chi_\eta$ does not depend on $I \setminus A$. It follows from our choice of the families $\{\chi_\eta : \eta < \omega_1\}$, $\{x_\eta : \eta < \omega_1\}$, and $\{J_\eta : \eta < \omega_1\}$ (see conditions (2)–(4) of Lemma 4.1) that $\text{supp}(x_\eta) \subseteq J_\eta \setminus \bigcup_{\zeta < \eta} J_\zeta$ and $\chi_\eta(x_\eta) \neq 1$, for each $\eta < \omega_1$. Since the sets $A_\eta = J_\eta \setminus \bigcup_{\zeta < \eta} J_\zeta$ are pairwise disjoint, there exists $\eta < \omega_1$ such that $A \cap A_\eta = \emptyset$. Since $\chi_\eta$ does not depend on $I \setminus A$, this implies that $\chi_\eta(x_\eta) = 1$, which is a contradiction.

We have thus proved that there exists a countable set $J \subset I$ such that $K$ does not depend on $I \setminus J$. Let $\varphi = (\pi_J|_G)^\wedge$. Then, by Lemma 4.8, $\varphi$ is a topological isomorphism of $(PD_J)^\wedge$ onto a closed subgroup of $G^\wedge$ containing $K$. Clearly, $L = \varphi^{-1}(K)$ is a compact subset of $(PD_J)^\wedge$ and $\varphi(L) = K$. Since the group $PD_J$ is reflexive, it contains, by Theorem 3.2, an open subgroup $U$ such that $L$ is constant on $U$. Then $K$ is constant on $G \cap \pi_J^{-1}(U)$ and, therefore, $G$ is reflexive. □

Theorem 4.9 makes it possible to find many proper dense reflexive subgroups of big products of (pseudo-)feathered groups endowed with the $P$-modified topology. This will be done in Corollary 4.12 with the help of the following result about permanence properties of the class of reflexive $P$-groups.

**Proposition 4.10.** Let $G$ be a dense subgroup of a topological group $H$. If $G$ is a reflexive $P$-group, so is $H$. 

Proof. Since $G$ is dense in $H$, [2, Lemma 4.4.1 (d)] implies that $H$ is a $P$-group as well. As indicated in Theorem 3.2, $\alpha_H$ is an open isomorphism. It is a well-known fact that if $G$ is a dense subgroup of a topological group $H$, then $\alpha_H$ is continuous provided so is $\alpha_G$. Indeed, if $K \subset H^\wedge$ is compact and $r$ denotes the natural restriction homomorphism of $H^\wedge$ to $G^\wedge$, then $r(K)$ is compact as well. By the continuity of $\alpha_G$, there is an open neighbourhood $U$ of the identity in $G$ such that $r(K) \subset r(U^\wedge)$. By denseness we have that $\overline{U}$ (the closure in $H$) is a neighbourhood of the identity in $H$, and since $K \subset U^\wedge = \overline{U}^\wedge$, the continuity of $\alpha_H$ follows.

In our proposition, $\alpha_G$ is continuous because $G$ is reflexive, so $\alpha_H$ is continuous. Therefore $H$ is a reflexive $P$-group. $\square$

The following corollary to Proposition 4.10 is immediate.

**Corollary 4.11.** Let $H$ be a non-reflexive $P$-group. Then no dense subgroup of $H$ is reflexive.

All reflexive $P$-groups constructed so far in Theorems 3.5 and 4.6 are complete since $P$-modifications preserve completeness in topological groups (see [14, Theorem 8]). It turns out, however, that reflexive $P$-groups need not be complete. To find examples of this phenomenon we need the following fact.

**Corollary 4.12.** Suppose that $D = \prod_{i \in I} D_i$ is a product of feathered Abelian groups and let

$$\Sigma_{\aleph_1} D = \{ x \in PD : |\text{supp}(x)| \leq \aleph_1 \}.$$  

Then every group $G$ satisfying $\Sigma_{\aleph_1} D \subset G \subset PD$ is reflexive.

**Proof.** According to Proposition 4.10 it suffices to show that $\Sigma_{\aleph_1} D$ is reflexive. It is clear that $\pi_J(\Sigma_{\aleph_1} D) = D_J = \prod_{i \in J} D_i$ for each $J \subset I$ with $|J| \leq \aleph_1$, where $\pi_J : D \to D_J$ is the projection. By Theorem 4.6, the groups $PD_J$ are reflexive. One applies Theorem 4.9 to conclude that the group $\Sigma_{\aleph_1} D$ is reflexive as well. $\square$

**Corollary 4.13.** There exist reflexive non-complete $P$-groups.

**Proof.** Let $F$ be a non-trivial discrete Abelian group and $D = F^\tau$ the product of $\tau$ copies of $F$, where $\tau > \aleph_1$. By Corollary 4.12, the subgroup $G = \Sigma_{\aleph_1} D$ of $PD$ is reflexive. It follows from $\tau > \aleph_1$ that $G$ is a proper dense subgroup of $PD$. Hence $G$ cannot be complete. $\square$
The subgroup $\Sigma_{\aleph_1}D$ of $PD = PF^\tau$ in the proof of the above corollary has character (meaning the minimal cardinality of a local base at the identity of $\Sigma_{\aleph_1}D$) greater than or equal to $\tau$. Since $\tau > \aleph_1$, all non-complete reflexive $P$-groups that arise due to Corollary 4.12 have character at least $\aleph_2$. It is a considerably more difficult job to find a non-complete reflexive $P$-group of the minimal possible character $\aleph_1$. We will do this in Corollary 4.15 below.

Let $\tau$ be an uncountable cardinal and $\mathbb{Z}_2 = \{1, -1\}$ the two-element (multiplicative) subgroup of $\mathbb{T}$. In the rest of this section we will denote by $\Pi$ the group $P\mathbb{Z}_2^\tau$ and by $\Sigma$ its subgroup $\Sigma\mathbb{Z}_2^\tau$.

Let $\xi$ be an arbitrary ultrafilter on $\tau$ containing all subsets of $\tau$ with countable complement. Note that $A \in \xi$ implies $|A| > \omega$. We then define

$$G_\xi = \{x \in \Pi : \text{supp}(x) \notin \xi\}.$$

It is straightforward to check that $G_\xi$ is a subgroup of $\Pi$. Also, if $x \in \Sigma$, then $|\text{supp}(x)| \leq \omega$. So, $x \notin \xi$ and therefore $\Sigma \subset G_\xi$. It is clear that $G_\xi \neq \Pi$ because the constant function $-1$ is not in $G_\xi$. Actually this is the “most important” element absent in $G_\xi$, as $\Pi = G_\xi \oplus (-1)$.

Let us show first that the subgroup $G_\xi$ of the $P$-group $\Pi = P\mathbb{Z}_2^\tau$ is reflexive in the special case when $\tau = \omega_1$.

**Proposition 4.14.** Every compact set $K \subset (G_\xi)^\wedge$ depends at most on countably many coordinates, where $\xi$ is an ultrafilter on $\omega_1$ containing the complements to countable sets. Hence the group $G_\xi$ is reflexive.

**Proof.** On the contrary, suppose that a compact set $K \subset (G_\xi)^\wedge$ depends on uncountably many coordinates. We construct three sets $\{\chi_\eta : \eta < \omega_1\} \subset K$, $\{J_\eta : \eta < \omega_1\} \subset [\omega_1]^{<\omega}$, and $\{x_\eta : \eta < \omega_1\} \subset G_\xi$ satisfying the following conditions for all $\eta < \omega_1$:

(i) $\chi_\eta$ does not depend on $\omega_1 \setminus J_\eta$;
(ii) $J_\zeta \subset J_\eta$ if $\zeta < \eta$;
(iii) $\eta \in J_\eta$;
(iv) $\text{supp}(x_\eta) \subset J_\eta \setminus \{\eta\}$;
(v) $\chi_\eta(x_\eta) = -1$;
(vi) if $\zeta < \eta$, then $J_\zeta \cap \text{supp}(x_\eta) = \emptyset$. 


Choose a character $\chi_0 \in K$ such that $\chi_0(y) = -1$ for some $y \in \Sigma$ with $0 \notin \text{supp}(y)$. Take a countable set $J_0 \subset \omega_1$ such that $0 \notin J_0$ and $\chi_0$ does not depend on $\omega_1 \setminus J_0$. Since $\Sigma \subset G_\xi$, we can find an element $x_0 \in G_\xi$ such that $\text{supp}(x_0) \subset J_0 \setminus \{0\}$ and $\chi_0(x_0) = -1$.

Suppose that for some $\eta < \omega_1$, the sequences $\{\chi_\zeta : \zeta < \eta\} \subset K$, $\{J_\zeta : \zeta < \eta\} \subset [\omega_1]^\leq \omega$, and $\{x_\zeta : \zeta < \eta\} \subset G_\xi$ have been defined to satisfy conditions (i)–(vi). Then we put $T_\eta = \bigcup_{\zeta < \eta} J_\zeta$ and choose $x_\eta \in K$ depending on the set $\omega_1 \setminus (T_\eta \cup \{\eta\})$. Such a choice of $x_\eta$ is possible since the set $T_\eta \cup \{\eta\}$ is countable and $K$ depends on uncountably many coordinates. Then pick $x_\eta \in G_\xi$ such that $\text{supp}(x_\eta) \cap (T_\eta \cup \{\eta\}) = \emptyset$ and $\chi_\eta(x_\eta) = -1$.

Let $J'_\eta$ be a countable subset of $\omega_1$ such that $\chi_\eta$ does not depend on $\omega_1 \setminus J'_\eta$. Then the set $J_\eta = J'_\eta \cup T_\eta \cup \text{supp}(x_\eta) \cup \{\eta\}$ is countable and $\chi_\eta$ does not depend on $\omega_1 \setminus J_\eta$. It is easy to see that the sets $\{\chi_\zeta : \zeta \leq \eta\}$, $\{J_\zeta : \zeta \leq \eta\}$, and $\{x_\zeta : \zeta \leq \eta\}$ satisfy (i)–(vi) at the step $\eta$.

For every $A \in \xi$, put $F_A = \left(\chi_\eta : \eta \in A\right)^{\mathbb{T}_{\mathbb{Z}}^{\omega_\xi}}$ and $\mathcal{F} = \{F_A : A \in \xi\}$. It follows from $\chi_\eta \in K$ for all $\eta < \omega_1$ and the compactness of $K$ that $F_A \subset K$, for each $A \in \xi$. Since $\mathcal{F}$ is a family of closed subsets of the compact space $\mathbb{T}_{\mathbb{Z}}^{\omega_\xi}$ with the finite intersection property, $\mathcal{F}$ has non-empty intersection. Let $\rho$ be a point in $\bigcap\{F_A : A \in \xi\}$. Clearly, $\rho \in K$, so $\rho$ is continuous. Let $J_\rho$ be a countable subset of $\omega_1$ such that $\rho$ does not depend on $\omega_1 \setminus J_\rho$.

Since $\Sigma$ is a dense subgroup of both groups $G_\xi$ and $\Pi = P\mathbb{Z}_2^{\omega_1}$, the characters $\rho$ and $\chi_\eta$ admit continuous extensions $\overline{\rho} : \Pi \rightarrow \mathbb{T}$ and $\overline{\chi}_\eta : \Pi \rightarrow \mathbb{T}$, for each $\eta < \omega_1$. Again, the density of $G_\xi$ in $\Pi$ implies that $\mathcal{F}$ does not depend on $\omega_1 \setminus J_\rho$ and $\mathcal{F}_\eta$ does not depend on $\omega_1 \setminus J_\eta$.

We recall that $-1$ is the element of $\Pi$ all of whose coordinates are equal to $-1$. For every $\eta < \omega_1$, let $H_\eta = \{x \in \Pi : x(\eta) = 1\}$ and take a character $\psi_\eta$ on $H_\eta = H_\eta \oplus (-1)$ defined by $\psi_\eta(x) = \overline{\chi}_\eta(x)$ and $\psi_\eta(-1 \cdot x) = -\overline{\chi}_\eta(x) \cdot \overline{\rho}(-1)$, for all $x \in H_\eta$. Here and in what follows $-1 \cdot x$ denotes the element of $\Pi$ defined by $(\eta \cdot \alpha)(\alpha) = \eta(\alpha^{-1})$, for each $\alpha < \omega_1$.

**Claim 1.** The character $\psi_\eta$ does not depend on $\omega_1 \setminus J_\eta$.

*Proof of Claim 1.* Since $\eta \in J_\eta$, we have $U(J_\eta) \subset H_\eta$. It then follows from the above definition that $\psi_\eta$ does not depend on $\omega_1 \setminus J_\eta$.

**Claim 2.** $\psi_\eta(x_\eta) = \chi_\eta(x_\eta) = -1$ for each $\eta < \omega_1$.

*Proof of Claim 2.* Indeed, $\eta \notin \text{supp}(x_\eta)$ by (iv) of the recursive construction, so $x_\eta \in H_\eta$ for each $\eta < \omega_1$. Hence the definition of $\psi_\eta$ and (v) together imply that $\psi_\eta(x_\eta) = \overline{\chi}_\eta(x_\eta) = \chi_\eta(x_\eta) = -1$.

**Claim 3.** For all $\alpha < \omega_1$, $\overline{\rho} \in \{\psi_\eta : \alpha \leq \eta < \omega_1\}^{\mathbb{T}_{\mathbb{Z}}^{\omega_\xi}}$. 

According to Theorem 4.9 it suffices to verify that the subgroup

\[ \pi(J) \] of the group \( P\mathbb{Z}_2^\omega \) is reflexive, for every set \( J \subset \tau \) satisfying \( |J| \leq \aleph_1 \). Let us consider two possible cases.

**Proof of Claim 3.** Fix \( \alpha < \omega_1 \) and take \( \{g_1, \ldots, g_n\} \subset \Pi \). We can assume that there exists \( m \leq n \) such that \( \{g_1, \ldots, g_m\} \subset G_\xi \) and \( \{g_{m+1}, \ldots, g_n\} \subset \Pi \setminus G_\xi \). Then \( \{g_1, \ldots, g_m, -1 \cdot g_{m+1}, \ldots, -1 \cdot g_n\} \subset G_\xi \). Let

\[
A = \omega_1 \setminus \bigcup_{i \leq m} \text{supp}(g_i), \quad B = \bigcap_{m < k \leq n} \text{supp}(g_k), \quad \text{and} \quad C = A \cap B \cap [\alpha, \omega_1).
\]

It follows from our choice of \( g_1, \ldots, g_n \) that \( C \in \xi \). So, \( \rho \in F_C \). Take \( \eta \in C \) such that \( \rho(g_i) = \chi_\eta(g_i) \) whenever \( 1 \leq i \leq m \) and \( \rho(-1 \cdot g_k) = \chi_\eta(-1 \cdot g_k) \) whenever \( m < k \leq n \). If \( 1 \leq i \leq m \), then \( g_i(\eta) = 1 \) because \( \eta \in A \). So, \( g_i \in H_\eta \) and \( \psi_\eta(g_i) = \chi_\eta(g_i) = \rho(g_i) = \pi(g_i) \). If \( m < k \leq n \), then \( (-1 \cdot g_k)(\eta) = 1 \) as \( \eta \in B \). So, \( -1 \cdot g_k \in H_\eta \) and \( \psi_\eta(-1 \cdot g_k) = \chi_\eta(-1 \cdot g_k) = \rho(-1 \cdot g_k) = \pi(-1 \cdot g_k) \). Since \( \psi_\eta \) and \( \pi \) are homomorphisms and \( \psi_\eta(-1) = \pi(-1) \), we see that \( \psi_\eta(g_k) = \pi(g_k) \). Therefore, \( \pi \in \{ \psi_\eta : \alpha \leq \eta < \omega_1 \}^{\pi^\omega} \). This completes the proof of Claim 3.

Now, we have a character \( \pi \in \Pi^\omega \), a family of characters \( \Psi = \{ \psi_\eta : \eta < \omega_1 \} \subset \Pi^\omega \), a family \( J = \{ J_\eta : \eta < \omega_1 \} \) of countable subsets of \( \omega_1 \), and a set \( X = \{ x_\eta : \eta < \omega_1 \} \subset G_\xi \). If follows from Claim 1, Claim 2, and the above conditions (iv) and (vi) that \( \Psi, J, \) and \( X \) satisfy (1)–(4) of Lemma 4.1 (with \( \psi_\eta \)'s in place of \( \chi_\eta \)'s). Since, by Claim 3, \( \pi \in \bigcap_{\alpha < \omega_1} \{ \psi_\eta : \alpha \leq \eta < \omega_1 \}^{\pi^\omega} \), Lemma 4.1 implies that \( \pi \) is discontinuous. This contradiction shows that the compact set \( K \) depends at most on countably many coordinates. The reflexivity of \( G_\xi \) now follows from Corollary 3.3.

It is easy to verify that the character (again, the minimum cardinality of a local base at the neutral element) of the group \( \Pi = P\mathbb{Z}_2^\omega \) is equal to \( \aleph_1 \). Since \( G_\xi \) is a proper dense subgroup of \( \Pi \), it cannot be complete. Hence Proposition 4.14 implies the following:

**Corollary 4.15.** There exists a reflexive non-complete \( P \)-group of the minimal possible character \( \aleph_1 \).

Let us extend the conclusion of Proposition 4.14 to subgroups \( G_\xi \) of the group \( P\mathbb{Z}_2^\tau \), for any cardinal \( \tau > \omega \).

**Theorem 4.16.** Let \( \tau > \omega \) be a cardinal and \( \xi \) an ultrafilter on \( \tau \) containing the complements to countable sets. Then the subgroup \( G_\xi \) of \( P\mathbb{Z}_2^\tau \) is reflexive.

**Proof.** According to Theorem 4.9 it suffices to verify that the subgroup \( \pi(J)(G_\xi) \) of the group \( P\mathbb{Z}_2^\xi \) is reflexive, for every set \( J \subset \tau \) satisfying \( |J| \leq \aleph_1 \). Let us consider two possible cases.
Case 1. \( J \not\in \xi \). Then the definition of \( G_\xi \) implies that \( \pi_J(G_\xi) = P\mathbb{Z}_2^J \), so the reflexivity of \( \pi_J(G_\xi) \) is immediate from Theorem 3.5.

Case 2. \( J \in \xi \). Put \( \eta = \{ J \cap A : A \in \xi \} \). Then \( \eta \) is an ultrafilter on \( J \) containing the complements to countable sets. Further, the definition of \( G_\xi \) implies that the projection \( \pi_J(G_\xi) \) of \( G_\xi \) coincides with the subgroup \( G_\eta \) of \( P\mathbb{Z}_2^J \). Identifying \( J \) and \( \omega_1 \) and applying Proposition 4.14, we see that the group \( \pi_J(G_\xi) \) is again reflexive. \( \square \)

5. Prodiscrete groups of countable pseudocharacter

As indicated in Theorem 3.1 metrizable prodiscrete groups are reflexive. We show in this section that metrizability cannot be relaxed to countability of the pseudocharacter.

Example 5.1. Every bounded torsion group \( G \) of cardinality \( \mathfrak{c} = 2^\omega \) admits a Hausdorff group topology \( \tau \) such that:

1. \((G, \tau)\) is prodiscrete but not discrete.
2. There is a countable family \( \{ H_n : n < \omega \} \) of open subgroups of \((G, \tau)\) such that \( \bigcap_{n < \omega} H_n = \{0\} \), i.e., \((G, \tau)\) has countable pseudocharacter.
3. \((G, \tau)^\wedge\) is discrete.

In particular, the group \((G, \tau)\) fails to be reflexive.

Proof. Let \( K = G^\ast \) denote the compact group that arises as the dual of \( G \) when the latter group carries the discrete topology. As \( \tau \) we consider the topology of uniform convergence on convergent sequences of \( K \). Denoting by \( \tau_b \) the maximal totally bounded group topology on \( G \), i.e., the topology of pointwise convergence on \( K \), we have that \( \tau_b \subseteq \tau \). It is well known that no infinite subset of \( G \) is \( \tau_b \)-compact [19]. Therefore \((G, \tau)\) does not contain infinite compact subsets either. Since all characters of \( G \) are \( \tau_b \)-continuous (and, a fortiori, \( \tau \)-continuous), we conclude that \( (G, \tau)^\wedge = K \), both algebraically and topologically. This proves (3).

As every compact group of weight \( \mathfrak{c} \), \( K \) is separable by [2, Corollary 5.2.7]. Let \( D = \{ \chi_n : n < \omega \} \) be a dense subset of \( K \) and consider, for each \( n < \omega \), the 1-element set \( S_n = \{ \chi_n \} \). Then \( S_n^\perp \) is an open subgroup of \((G, \tau)\) and \( \bigcap_{n < \omega} S_n^\perp = \{0\} \). Assertion (2) is thus proved.

Given \( S \subseteq K \) and \( k > 0 \), we define

\[ S^{\bullet,k} = \{ g \in G : \chi(g) = e^{it} \text{ with } |t| < 1/k, \text{ for every } \chi \in S \} \]
Assume that $n$ is the exponent of $G$ and choose $k$ such that the set $\{e^{it} : |t| < 1/k\}$ does not contain $n$-roots of the identity different from 1. The collection $\{S^{\circ,k} : S \text{ is a convergent sequence in } K\}$ is then a subbasis of neighbourhoods of the identity of $(G, \tau)$. Since $\chi(G)$ consists of $n$-roots of the identity, $S^{\circ,k} = S^\perp = \{g \in G : \chi(g) = 1 \text{ for every } \chi \in S\}$, and $S^{\circ,k}$ is an open subgroup of $(G, \tau)$. Therefore $(G, \tau)$ is protodiscrete but $(G, \tau)$ cannot be discrete because, by [9, Corollary 2.3], a closed subset $L$ of $K$ with $L^\perp = \{0\}$ must have cardinality $|L| \geq \tau$.

To prove (1) we must check that $(G, \tau)$ is complete. To this end, let $L = \{x_\lambda : \lambda \in \Lambda\}$ be a Cauchy sequence in $(G, \tau)$. Regarding elements of $G$ as continuous homomorphisms of $K$ into $T$, we have that $L$ is also a Cauchy sequence in $T^K$. There is therefore $X_0 : K \to T$ such that $X_0$ is the pointwise limit of $x_\lambda$'s. Since the property of being a homomorphism is preserved by pointwise limits, $X_0$ is a homomorphism. We now check that $X_0$ is indeed continuous. If $S$ is a convergent sequence in $K$, then $x_\lambda|_S$ converges uniformly to $X_0|_S$, and, therefore, $X_0|_S$ is continuous. So $X_0$ is a sequentially continuous homomorphism of $K$ to $T$. By a well-known theorem of Varopoulos [21], sequentially continuous homomorphisms of compact groups (provided they are of Ulam-nonmeasurable cardinality) are continuous, hence $X_0 \in K^\wedge = G$. Thus $X_0 \in G$ and $\lim x_\lambda = X_0$ pointwise. Since $(x_\lambda)_\lambda$ is a Cauchy net in $(G, \tau)$ it follows that $x_\lambda$ must be $\tau$-convergent to $X_0$. Therefore $(G, \tau)$ is complete. □

6. Determined $P$-groups

Following [8], we say that a subgroup $G$ of a topological Abelian group $H$ determines $H$ if the restriction mapping $\chi \mapsto \chi|_G$ is a topological isomorphism of $H^\wedge$ onto $G^\wedge$. If every dense subgroup of $H$ determines it, we say that $H$ is determined. It was established independently by Außenhoffer [3] and Chasco [7] that every metrizable Abelian group is determined.

In compact non-metrizable groups, dense subgroups can or can not determine the whole group. However, every determined compact group is metrizable [13] (see also [9]).

We show below that no proper dense subgroup of a $P$-group determines the whole group. This indicates once again the difference between the dual properties of the class of $P$-groups on one hand and the classes of compact or metrizable groups on the other hand.

**Proposition 6.1.** If a dense subgroup $G$ of a $P$-group $H$ determines $H$, then $G = H$. 
Proof. Suppose that a dense subgroup \( G \subset H \) determines \( H \). Then the restriction mapping \( r: H^\wedge \to G^\wedge \) defined by \( r_G(\chi) = \chi|_G \) for each \( \chi \in H^\wedge \), is a topological isomorphism. Hence the dual homomorphism \( r^\wedge: G^{\wedge\wedge} \to H^{\wedge\wedge} \) is also a topological isomorphism, where \( r^\wedge \) is defined by \( r^\wedge(\lambda) = \lambda \circ r \in H^{\wedge\wedge} \) for each \( \lambda \in G^{\wedge\wedge} \).

It is easy to see that the mappings \( \alpha_H, \alpha_G, \) and \( r^\wedge \) satisfy the equality \( \alpha_H|_G = r^\wedge \circ \alpha_G \). Indeed, given any \( x \in G \) and \( \chi \in H^\wedge \), we have that

\[
((r^\wedge \circ \alpha_G)(x))(\chi) = r^\wedge(\alpha_G(x))(\chi) = \alpha_G(x)(\chi|_G) = \chi(x) = ((\alpha_H|_G)(x))(\chi).
\]

Since \( x \in G \) is arbitrary, we conclude that \( r^\wedge \circ \alpha_G = \alpha_H|_G \).

Take an arbitrary element \( x \in H \). According to Theorem 3.2, the evaluation mapping \( \alpha_G: G \to G^{\wedge\wedge} \) is surjective. In addition, since \( r^\wedge(G^{\wedge\wedge}) = H^{\wedge\wedge} \), there exists \( y \in G \) such that \( \alpha_G(y) = (r^\wedge)^{-1}(\alpha_H(x)) \). Therefore, \( r^\wedge(\alpha_G(y)) = \alpha_H(x) \). It now follows from the equality \( r^\wedge(\alpha_G(y)) = \alpha_H(y) \) that \( \alpha_H(x) = \alpha_H(y) \), and the injectivity of \( \alpha_H \) implies that \( x = y \). This proves that \( x \in G \) and, therefore, \( G = H \). \( \square \)

7. Open problems

Here we present several problems whose solutions can substantially improve our understanding of the duality theory for \( P \)-groups.

One can try to generalize Proposition 4.4 as follows:

**Problem 1.** Suppose that \( G \) is a pseudocompact reflexive group. Is the \( P \)-modification \( PG \) of \( G \) reflexive? What if \( G \) is precompact?

Since, by [12, Lemma 4.1], the dual group \( G^\wedge \) is \( \omega \)-bounded (hence pseudocompact) for every \( P \)-group \( G \), the following problem is a special case of Problem 1:

**Problem 2.** Let \( G \) be a reflexive \( P \)-group. Is the \( P \)-modification of the dual group \( G^\wedge \) reflexive?

By Proposition 6.1, the next problem is equivalent to asking whether there exists a non-discrete \( P \)-group without proper dense subgroups:

**Problem 3.** Does there exist a non-discrete determined \( P \)-group?
Problem 4. Does every non-discrete reflexive $P$-group contain a proper dense reflexive subgroup?

The next two problems are motivated by Proposition 4.10 and Corollary 4.11:

Problem 5. Let $\kappa$ be an uncountable cardinal. Does the group $P\mathbb{Z}_2^\kappa$ contain a dense reflexive subgroup $G$ such that every proper dense subgroup of $G$ is not reflexive?

Notice that if the answer to Problem 4 is “yes”, then the answer to Problem 5 is “no”.

Problem 6. Let $\kappa$ be an uncountable cardinal. Does the group $P\mathbb{Z}_2^\kappa$ contain a dense non-reflexive subgroup $G$ such that every group $H$ with $G \subseteq H \subseteq P\mathbb{Z}_2^\kappa$ is reflexive?

Clearly, one can formulate more general versions of Problems 5 and 6 by replacing $\mathbb{Z}_2^\kappa$ with an arbitrary compact non-metrizable group $K$ and taking dense subgroups of $PK$.

According to Corollary 4.15, there exists a non-complete reflexive $P$-group of character $\aleph_1$. We do not know if character can be replaced with weight in this result:

Problem 7. Does there exist in ZFC a non-complete reflexive $P$-group of weight $\aleph_1$?

A topological group $G$ is called $\omega$-narrow if $G$ can be covered by countably many translates of every neighbourhood of the identity (see [2, Section 3.4]). The groups with this property constitute one of the most important classes in the theory of topological groups. A direct verification shows that all reflexive $P$-groups given by Theorems 3.5 and 4.6 fail to be $\omega$-narrow.

Problem 8. Do there exist non-discrete $\omega$-narrow reflexive $P$-groups?

In the next problem we pretend to generalize Theorem 4.16:

Problem 9. Let $\tau$ be an uncountable cardinal and $G$ be a subgroup of $\Pi = P\mathbb{Z}_2^\tau$ such that $|\Pi/G| < \omega$ (or $|\Pi/G| \leq \omega$). Is $G$ reflexive? What if, additionally, $G$ contains $\Sigma\Pi$?

It was mentioned in Section 6 that a compact group is determined if and only if it is metrizable. We do not know if this result can be extended to pseudocompact groups:

Problem 10. Is every determined pseudocompact Abelian group metrizable?

Example 5.1 shows that prodiscrete groups of countable pseudocharacter may be non-reflexive. Algebraically, our examples are bounded torsion groups. We do not know whether this condition is necessary:
Problem 11. Does \( \mathbb{Z}^c \) contain any closed subgroup of countable pseudocharacter that is not reflexive?

By Proposition 6.1, \( P \)-groups admitting proper dense subgroups cannot be determined. It is not clear how prodiscrete groups of countable pseudocharacter behave in this regard.

Problem 12. Are prodiscrete groups of countable pseudocharacter determined?

References


Jorge Galindo, Instituto de Matemáticas y Aplicaciones (IMAC), Departamento de Matemáticas, Universidad Jaume I, E-12071, Castellón, Spain.

E-mail: jgalindo@mat.uji.es

Luis Recoder-Núñez, Department of Mathematical Sciences, Central Connecticut State University, 1615 Stanley Street, New Britain, CT 06050, USA.

E-mail: recoderl@ccsu.edu

Mikhail Tkachenko, Departamento de Matemáticas, Universidad Autónoma Metropolitana, Av. San Rafael Atlixco 186, Col. Vicentina, Iztapalapa, C.P. 09340, México, D.F., Mexico.

E-mail: mich@xanum.uam.mx