UNIVERSITAT JAUME I

Tesi Doctoral

Creixement del volum i nombre de finals de subvarietats

Autor:

Vicent Gimeno Garcia

Director:

Vicente Palmer Andreu

Tesi presentada per optar al Grau de Doctor

Castelló de la Plana, Juliol del 2012

A Laia i Adrià.

Tôt ou tard les âges héroïques de l'exploration des montagnes prendront fin comme ceux de l'exploration de la planète elle-même, et le souvenir des fameux gravisseurs se transformera en légende. Les unes après les autres, toutes les montagnes des contrées populeuses auront été escaladées: des sentiers faciles, puis des chemins carrossables, auront été construits de la base au sommet, pour en faciliter l'accès, même aux desoeuvrés et aux affadis ...

ÉLISÉE RECLUS, 1880.

Why did the chicken cross the road?
To get to the other side.

Common joke.

Why did the chicken cross the Mobius Strip?
To get to the same side!

SHELDON COOPER.

Agraïments

Aquesta tesi representa el final d'un procés d'aprenentatge i de recerca que ha donat els fruits que ací es presenten. Però aquesta tesi no és sols els resultats publicats, és també tot allò que no està publicat i que ha segut important durant tot aquest procés, com són les converses amb Jorge Galindo del departament de matemàtiques de la UJI i que em van ser molt útils per a descartar tota una sèrie d'idees errònies que m'hagueren fet perdre molt de temps.

Part de la tesi és també l'assessorament matemàtic que em va donar Juanjo Nuño de la Universitat de València i malgrat que no s'ha traduït en cap resultat per a aquesta tesi va contribuir en el seu moment a la clarificació de conceptes importants per a poder seguir endavant.

Tesi, també és el suport que he rebut per part del departament de matemàtiques de la UJI, i tots els professors i professores que en major o menor mesura m'han recolzat en la meua recerca.

Menció especial es mereix el meu director, Vicent Palmer pel temps que m'ha dedicat, pels esforços realitzats, per la confiança que ha dipositat en mi, i per tot allò que m'ha ensenyat durant aquest període. I sobretot, per la paciència mútua que hem sabut tindre l'un amb l'altre. Vicent Palmer m'ha ensenyat un ofici, el de la recerca matemàtica, des de la vessant científica i des de la vessant humana amb la seua bonhomia i proximitat.

En la llista de coses que constitueixen una tesi més enllà dels resultats acadèmics també figura malauradament totes les amistats, familiars, companys i companyes en general, als quals no els he pogut dedicar l'atenció necessària durant tot aquest temps. El meu més sentit agraïment per la seua comprensió.

Organització de la Tesi

Aquesta Tesi Doctoral compta de dues parts diferenciades: per una part, els capítols **??** i 2 que serveix com a introducció i resum dels resultats i conceptes previs que s'utilitzen al llarg de tot el treball. I els capítols 3, 4, 5, on presentem tots els resultats originals.

a) El capítol 3 és l'article EXTRINSIC ISOPERIMETRY AND COMPACTIFICATION OF MINIMAL SURFACES IN EUCLIDEAN AND HYPERBOLIC SPACES. Vicent Gimeno & Vicente Palmer. *arXiv:1011.5380v2* **2010** i acceptat per a la seua publicació en *Israel Journal of Mathematics.*

b) El capítol 4 és l'article de VOLUME GROWTH AND THE CHEEGER ISOPERIMETRIC CONSTANT OF SUBMANIFOLDS IN MANIFOLDS WHICH POSSES A POLE. Vicent Gimeno & Vicente Palmer.*arXiv:1104.5625v2* **2011** i acceptat per a la seua publicació en *Proceedings of the American Mathematical Society.*

c) El capítol 5 és l'article COMPLETE SUBMANIFOLDS WITH FINITE TOPOLOGY AND GAP PHENOMENONS FOR SUBMANIFOLDS AND MANIFOLDS. Vicent Gimeno & Vicente Palmer *arXiv: 1112.4042v2* **2011**.

Els capítols 3, 4, 5 són una còpia fidedigna de la versió publicada en el repositori de la universitat de Cornell arXiv.org, conservant la seua numeració de pàgines i la seua bibliografia.

Índex

1 Summary and results **1**
- 1.1 Subject of study and objectives . . . 1
- 1.2 Original contributions, first look . . . 1
- 1.3 Approach and methodology . . . 2
- 1.4 Topological type, number of ends and the Cheeger constant . . . 4
  - 1.4.1 Finite topological type and number of ends . . . 4
  - 1.4.2 Cheeger isoperimetric constant . . . 6
- 1.5 Previous results . . . 6
- 1.6 Original results and conclusions . . . 8
- 1.7 Further research lines . . . 10

2 Anàlisi amb la distància extrínseca **11**
- 2.1 Introducció . . . 11
- 2.2 Operadors diferencials en varietats de Riemann . . . 12
- 2.3 Operadors diferencials i immersions . . . 14
- 2.4 Funcions, punts crítics, teoria de Morse, teorema de Sard i formula de la coàrea . . . 19
- 2.5 Varietats amb pol . . . 22
- 2.6 Distància extrínsica i boles extrínseques . . . 25
- 2.7 Varietats producte i espais models . . . 27
- 2.8 Comparacions per al hessià: el teorema de Greene-Wu . . . 39
- 2.9 Comparacions per al volum i el diametre . . . 41

3 Isoperimetria extrínseca i compactificació **45**

4 Creixement del volum i constant de Cheeger **57**

5 Creixement del volum i nombre de finals **71**

Bibliografia **93**

1
Summary and results

1.1 Subject of study and objectives

Riemannian geometry is closely related to topology and analysis, this relationship being one of the main engines driving recent research in the field of geometric analysis, where the problems studied involve the convergence of differential equations and differential geometry.

This work is carried out within the following context: let us consider an isometric immersion $\varphi : P^m \to N^n$ from the manifold P to the manifold N. Then, we impose geometric restrictions on the ambient manifold N and on the immersion. The geometric restrictions on the manifold N are: the existence of a pole in N and the existence of a bound in its radial curvatures. And with regard to the immersion, these restrictions primarily affect its mean curvature.

In this context, we will try to obtain as much information about the geometry and topology of the submanifold P as possible. This will mainly refer to its *topological type*, the *number of ends* $\mathcal{E}(P)$, the *Cheeger constant* $\mathcal{I}_\infty(P)$ and the relationships among these concepts.

1.2 Original contributions, first look

As a first approach to understanding what the scope of this thesis is, we set out the following two theorems for minimal immersions in the Hyperbolic space that make use of almost all our results.

Theorem 1.2.1. *Let $\varphi : P^m \to \mathbb{H}^n$ be a minimal and complete immersion from the m-dimensional manifold P^m to the Hyperbolic space $\mathbb{H}^n(b)$ of constant sectional curvature $b < 0$. Suppose, moreover, that $m > 2$, P is properly immersed and $\|B^P\|(x) \leq \frac{\delta(r(x))}{e^{2\sqrt{-b}r(x)}}$, for all $x \in P$. Here, $\|B^P\|$ is the norm of the second fundamental form, δ is a positive function such that $\lim_{r\to\infty} \delta(r) = 0$ and r is the extrinsic distance. Then*

1. *P is of finite topological type.*
2. *$\mathcal{I}_\infty(P) = (m-1)\sqrt{-b}$*
3. *$\lim_{t\to\infty} \frac{\mathrm{Vol}(\varphi(P)\cap B_t^{b,n})}{\mathrm{Vol}(B_t^{b,m})} \leq \mathcal{E}(P)$, $B_t^{b,m}$ being the geodesic ball of radius t in $\mathbb{H}^m(b)$.*
4. *If P has only one end, P is a totally geodesic submanifold of $\mathbb{H}^n(b)$.*

Theorem 1.2.2. *Let $\varphi : P^2 \to \mathbb{H}^n$ be a minimal immersion from the complete surface P to the Hyperbolic space $\mathbb{H}^n(b)$ of constant sectional curvature $b < 0$. Suppose, moreover, that P has finite total extrinsic curvature , i.e. $\int_P \|B^P\|^2 d\sigma < \infty$. Then:*

1. *P is of finite topological type.*
2. *$\mathcal{I}_\infty(P) = \sqrt{-b}$*
3. $\lim_{t\to\infty} \frac{\mathrm{Vol}(\varphi(P)\cap B_t^{b,n})}{\mathrm{Vol}(B_t^{b,2})} \leq \frac{1}{4\pi}\int_P \|B^P\|^2 d\sigma + \chi(P)$.
4. *If $\frac{1}{4\pi}\int_P \|B^P\|^2 d\sigma + \chi(P) = 1$, P is a totally geodesic submanifold of $\mathbb{H}^n(b)$, where $\chi(P)$ denotes the Euler characteristic of P.*

It should be noted that for minimal surfaces in $\mathbb{R}^3$ with all ends embedded, thereby satisfying the hypothesis of the above theorem ($\int_P \|B^P\|^2 d\sigma < \infty$), we get that $\mathcal{E}(P) = \frac{1}{4\pi}\int_P \|B^P\|^2 d\sigma + \chi(p)$.

1.3 Approach and methodology

The main tools used in this study are the *extrinsic distance* and *extrinsic balls*, the construction of a rotationally symmetric space called *W-model comparison space*, the *W-volume growth*, and the comparisons with the *Hessian of the extrinsic distance* in manifold P and the Hessian of the usual distance in the $W-$model.

Regarding our tools, the extrinsic distance is the restriction by the immersion of the Riemannian distance of the manifold N to the manifold P. Since N is a manifold with a pole, the extrinsic distance becomes a function C^∞ on P (with the possible exception of the point we consider as the source in order to measure the distance). With this extrinsic distance we can define an extrinsic ball D_R of radius R as the set of all points in P that are at an extrinsic distance below the value of R (see Section 2.6 and Figure 1.1). From Sard's theorem and the inverse function theorem we know that the boundaries ∂D_R are smooth submanifolds of P for almost every value of R.

On the other hand, a $W-$model comparison space is a geometric construction called *model space*, which is a generalisation of surfaces of revolution (see Section 2.7) that is performed using a function W, which in our case depends on the restraints imposed on the radial curvatures of the manifold N and the radial mean curvature of the immersion. Finally, using the results of R. Greene and S. Wu ([GW79]), we can compare the Hessian (see section 2.8) of the extrinsic distance function on P and the Hessian of the usual distance function in the $W-$ model comparison space.

With these comparisons for the Hessian we can study what similarities and what differences there are between the extrinsic ball D_R of radius R and the geodesic ball $B_R^{M_W^m}$, also of radius R, in the $W-$model comparison space M_W^m.

When the immersion is proper, the extrinsic balls are precompact sets (see section 2.6) and we can make a comparison between the volume of the extrinsic ball D_R of radius R and the geodesic ball $B_R^{M_W^m}$ in model space M_W^m. Expressing this as a ratio:

$$\mathcal{Q}_W(R) := \frac{\mathrm{Vol}(D_R)}{\mathrm{Vol}(B_R^{M_W^m})}. \tag{1.3.1}$$

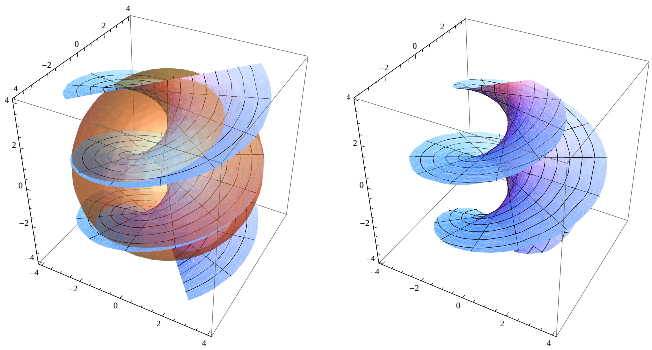


Figura 1.1: This graph represents a helicoid in $\mathbb{R}^3$, where the helicoid would play the role of the submanifold and $\mathbb{R}^3$ would play the role of the ambient manifold. The extrinsic ball of radius R would be the set of points on the helicoid that are at a distance below R in $\mathbb{R}^3$, as is shown in the enlarged figure on the right.

What is the behaviour of $\mathcal{Q}_W(R)$? Is this function increasing, decreasing or oscillating? This study is what what we call a *study of the $W-$volume growth*. When the immersion is proper and P is a complete and non-compact manifold, R ranges in $R \in (0, \infty)$ and we can study the asymptotic behaviour of $\mathcal{Q}_W(R)$. In particular, when $\mathcal{Q}_W(R)$ is an increasing function in R we will say that the immersion has *finite $W-$volume growth* if:

$$\lim_{R\to\infty} \mathcal{Q}_W(R) < +\infty. \tag{1.3.2}$$

Otherwise ($\lim_{R\to\infty} \mathcal{Q}_W(R) = +\infty$) we will say that the immersion has an *infinite W-volume growth*.

The study of the comparisons for the volume of geodesic balls or extrinsic balls is a classic area in the field of Riemannian geometry, and the following theorems can be cited with regard to geodesic balls: P. Günter, R.L. Bishop and M. Gromov.

Theorem 1.3.1. *(See [Gün60], [BC64], [Cha93]). Let M^n be a Riemannian manifold and let us also suppose that the sectional curvatures of M are all less than or equal to b. Then, for any $x \in M$, $B_R^M(x)$ being the geodesic ball of radius R centred at x, we have:*

$$\operatorname{Vol}\left(B_R^M(x)\right) \geq \operatorname{Vol}\left(B_R^{b,n}\right)$$

for any R less than the injectivity radius or less than $\pi/\sqrt{b}$ (when $b > 0$). If we have $\operatorname{Vol}\left(B_R^M(x)\right) = \operatorname{Vol}\left(B_R^{b,n}\right)$ for a certain radius, then the ball $B_R^M(x)$ is isometric to the ball $B_R^{b,n}$. Moreover, the function $f : \mathbb{R}^+ \to \mathbb{R}^+$, given by:

$$f(R) := \frac{\operatorname{Vol}\left(B_R^M(x)\right)}{\operatorname{Vol}\left(B_R^{b,n}\right)},$$

is an increasing function of R for all R below the injectivity radius or less than $\pi/\sqrt{b}$ (when $b > 0$).

Theorem 1.3.2. *(See [CGT82], Gromov's appendix in [MS86], [BC64], [Cha93]). Let M^n be a Riemannian manifold and let us suppose moreover that the Ricci curvatures of M are all greater than or equal to $(n-1)b$. Then, for any $x \in M$, we have:*

$$\operatorname{Vol}\left(B_R^M(x)\right) \leq \operatorname{Vol}\left(B_R^{b,n}\right),$$

and if for any R, $\operatorname{Vol}\left(B_R^M(x)\right) = \operatorname{Vol}\left(B_R^{b,n}\right)$, the ball $B_R^M(x)$ is isometric to the ball $B_R^{b,n}$. And the function $f:\mathbb{R}^+ \to \mathbb{R}^+$, given by:

$$f(R) := \frac{\operatorname{Vol}\left(B_R^M(x)\right)}{\operatorname{Vol}\left(B_R^{b,n}\right)},$$

is a decreasing function of R.

For extrinsic balls of minimal submanifolds in real space forms, we have the following:

Theorem 1.3.3. *(see[And82], [MP11], [Pal99]). Let P^m be a proper and minimal submanifold in a real space form $\mathbb{K}^n(b)$ of constant sectional curvature $b \leq 0$, then,*

$$\operatorname{Vol}(D_R) \geq \operatorname{Vol}\left(B_R^{b,m}\right), \quad \frac{\operatorname{Vol}(\partial D_R)}{\operatorname{Vol}(D_R)} \geq \frac{\operatorname{Vol}\left(S_R^{b,m-1}\right)}{\operatorname{Vol}\left(B_R^{b,m}\right)}.$$

If we have equality in either of the above inequalities, D_R is a minimal cone in $\mathbb{K}^n(b)$ (and when $b<0$, which implies that P is a totally geodesic submanifold of $\mathbb{K}^n(b)$).

Moreover, the function $f:\mathbb{R}^+ \to \mathbb{R}^+$ given by:

$$f(R) := \frac{\operatorname{Vol}(D_R)}{\operatorname{Vol}\left(B_R^{b,m}\right)},$$

is an increasing function of R.

1.4 Topological type, number of ends and the Cheeger constant

1.4.1 Finite topological type and number of ends

With regard to the objects of our study, the first will be the topological type of P. A manifold is of *finite topological type* when there is a homeomorphism to the interior of a compact manifold with boundary; otherwise we will say that the manifold is of *infinite topological type*.

Another object of study is the number of ends. The number of ends of a non-compact manifold P is a topological invariant that we can define, following [Tka94], as the smallest natural number $\mathcal{E}(P)$ such that for any compact set $C \subset P$ the number of connected non-compact components of $P \setminus C$ is less than or equal to $\mathcal{E}(P)$.

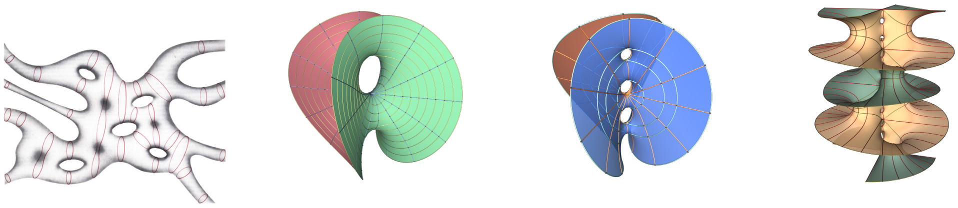


Figura 1.2: From left to right: Example of a surface where, if we imagine that the branches extend to infinity, it would have a finite number of ends (each branch would be an end) and finite genus (the genus would be the total number of holes). Part of the Chen-Gackstatter surface (genus 1 and one end, finite topology). Part of the Chen-Gackstatter with genus 3. Part of the Callahan-Hoffman-Meeks periodic surface, infinite genus and infinite number of ends.

To count the number of ends of P we can use any exhaustion $\{K_i\}_{i=1}^{\infty}$ with compacts $K_i \subset P$. Hence, we have one end in each sequence:

$$U_1 \supset U_2 \supset U_3 \subset \cdots \tag{1.4.1}$$

where U_j is a connected component of $P \setminus K_j$. The number of ends does not depend on the exhaustion.

In a proper immersion we use exhaustion by extrinsic balls to count the number of ends.

Let Σ be a connected, non-compact, $2-$dimensional manifold (i.e. a non-compact surface). If Σ is of finite topological type, by definition Σ is homeomorphic to the interior of a compact surface with boundary Σ_2. The boundary $\partial\Sigma_2$ of Σ_2 has a finite number $k \geq 1$ of connected components due to Σ_2 being compact. Since each connected component of $\partial\Sigma_2$ is a compact, connected and $1-$dimensional manifold, each connected component of the boundary is homeomorphic to $\mathbb{S}^1$. Following the method given in [Mas77] we can construct a compact surface without a boundary Σ_2^* by adding a closed disk to each connected component of $\partial\Sigma_2$. Hence, finally we can state, as in [Mas77], that Σ_2 (and therefore Σ) is homeomorphic to $\Sigma_2^* - \{p_1, \cdots, p_k\}$ (i.e. homeomorphic to a compact manifold Σ_2^* punctured with a finite number of points, one on each disk that is added). On the other hand, since Σ is homeomorphic to the interior of a compact set, Σ is of finite genus (recall that it can be shown, see [Mas77], that Σ_2^* is homeomorphic to the connected sum of n tori, and the genus of Σ_2^* (and of Σ) can be defined as this number n).

It is not hard to see that the number of ends of Σ is the number of connected components of the boundary of Σ_2, which also coincides with the number of points punctured in the compact surface without boundary Σ_2^*. Thus, a finite topological type surface has a finite number of ends and is of finite genus (see Figure 1.2).

Since the finite topological type surfaces are homeomorphic to compact surfaces punctured with a finite number of points, we can use the following definition for the number of ends: the number of ends of a non-compact surface Σ of finite topological type is the number of points that we have to puncture from a compact surface Σ^*

in order to make Σ^* homeomorphic to Σ. This definition of the number of ends is totally equivalent to the one stated above (see [Tka94]).

1.4.2 Cheeger isoperimetric constant

The last object of this study will be to analyse the numerical value of the *Cheeger isoperimetric constant* of the manifold P given with the geometric restrictions on the manifold N and on the mean radial curvature of the immersion $\varphi : P \to N$. The Cheeger constant $\mathcal{I}_\infty(P)$ of a non-compact manifold P (see Chapter 4) is defined as:

$$\mathcal{I}_\infty(P) = \inf_{\Omega \subset P} \frac{\text{Vol}(\partial\Omega)}{\text{Vol}(\Omega)}, \tag{1.4.2}$$

where Ω ranges in the compact domains $\Omega \subset P$ with smooth boundaries $\partial\Omega$.

1.5 Previous results

To list some of the most important results obtained prior to this work, we can focus on minimal submanifolds P^m of $\mathbb{R}^n$. In this setting there is a strong restriction on the mean curvature of the immersion (in fact it is 0 for any point in P) and there is also a strong restraint for the radial sectional curvatures of $\mathbb{R}^n$ (which are all 0 at any point in $\mathbb{R}^n$). In this setting, our model comparison space will be $\mathbb{R}^m$.

From the results of [And84], [JM83], [CO84], [CO67] and [CheQi95], we know that:

Theorem 1.5.1. *(see [And84], [JM83], [CO84], [CO67] and [CheQi95]) Let P^m be a complete minimal submanifold immersed in $\mathbb{R}^n$ with finite extrinsic curvature (i.e. $\int_P \|B^P\|^m < +\infty$, where $\|B^P\|$ is the norm of the second fundamental form), then:*

1. *P is properly immersed in $\mathbb{R}^n$.*
2. *The immersion is of finite volume growth.*
3. *P is of finite topological type (in particular it has a finite number of ends).*
4. *If either $m \geq 3$, or $m = 2$, $n = 3$ and each end of P is embedded, then*

$$\lim_{t\to\infty} \frac{\text{Vol}(D_t)}{V_m t^m} = \mathcal{E}(P) \tag{1.5.1}$$

where $\mathcal{E}(P)$ is the finite number of ends of P, D_t is the extrinsic ball of radius t and V_m is the volume of the geodesic ball of unit radius in $\mathbb{R}^m$.

Obviously, our theorem **??** (and part of the results shown in Chapter 5) are a generalisation of the above theorem.

The volume growth is closely related with the number of ends, as we can see in the following theorem.

Theorem 1.5.2. *(see [CheQi95]) Let P be a complete, proper and $m-$dimensional minimal submanifold of $\mathbb{R}^n$. Suppose that:*

$$Sup_{t>0} \frac{\text{Vol}(D_t)}{V_m t^m} < +\infty. \tag{1.5.2}$$

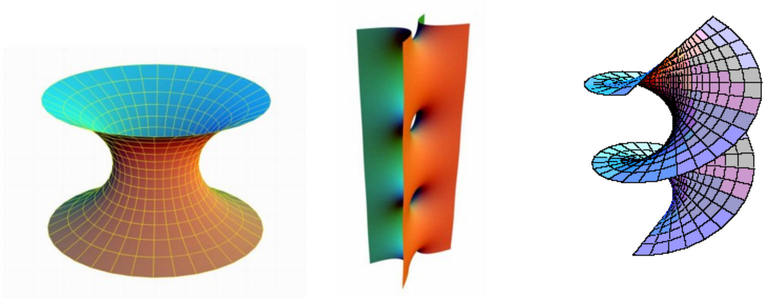


Figura 1.3: From left to right: catenoid (minimal surface with finite total curvature, finite topological type, finite volume growth and finite number of ends), single periodic Scherk surface (minimal surface with infinite total curvature, infinite topological type, finite volume growth and finite number of ends) and the helicoid (minimal surface with infinite total curvature, infinite volume growth, finite topological type and finite number of ends).

Then, for the number of ends of P (called $\mathcal{E}(P)$)

$$\mathcal{E}(P) \leq Sup_{t>0} \frac{\mathrm{Vol}(D_t)}{V_m t^m}, \tag{1.5.3}$$

where D_t is the extrinsic ball of radius t and V_m is the volume of the geodesic ball of unit radius in $\mathbb{R}^m$.

On the other hand, the topological type and the finite volume growth are closely related with the total curvature of the surfaces by the following theorem:

Theorem 1.5.3. *(see [CheQi97]) Let P be a complete, oriented surface immersed in $\mathbb{R}^n$; hence, if P is of finite topological type and finite volume growth, P has finite total curvature.*

Our theorem 1.2.2 (see Chapter 3) is based on the unified proof of the Cher-Osserman inequalities for Hyperbolic space, linking extrinsic total curvature with finite topological type.

Although from the above theorems it may seem that the total curvature, the topological type, the volume growth and the number of ends share the same finite or infinite type, this is not really the case. We can see from the examples in Figure 1.3 that the finiteness of some of the previous concepts does not necessarily imply the finiteness of all the others.

On the other hand, we can explore the influence of curvature on the geometric and topological properties of a complete Riemannian manifold. Classical results concerning this are the gap theorems showed by Greene and Wu in [GW82a]. Roughly speaking, Greene and Wu's results state that a Riemannian manifold with a pole and with faster than quadratic decay of its sectional curvatures is isometric to the Euclidean space. More examples concerning submanifolds immersed in an ambient Riemannian manifold and the analysis of their (intrinsic and extrinsic) curvature behaviour are the (Bernstein-type) gap results given by Kasue and Sugahara in [KS87] (see Theorems A and B), where an accurate (extrinsic) curvature decay forces the minimal (or not) submanifolds with one single end in the Euclidean and Hyperbolic spaces to be totally geodesic.

We obtained these “gap” results in property (4) of theorems **??** and 1.2.2, and with greater generality in Chapter 5.

1.6 Original results and conclusions

In Chapter 3 we have proved that for minimal surfaces immersed in $\mathbb{R}^n$ or in $\mathbb{H}^n(b)$ the finiteness of the total curvature implies the finiteness of the topological type and the finiteness of the volume growth. This results in unified proof of the Chern-Osserman inequalities for minimal surfaces in Hyperbolic and Euclidean space.

In Chapter 4 we study the relation between the finite volume growth and the Cheeger constant.

In Chapter 5 we have proved that with accurate decay of the norm of the second fundamental form the submanifold is properly immersed, is of finite topological type, and the number of ends is related with the finite volume growth.

These results may be understood in terms of the “rigidity” of these submanifolds with controlled extrinsic curvature. Here, we see that under appropriate geometric restraints, an immersion $\varphi : P \to N$, knowing certain geometric or analytic properties of a domain $\Omega \subset P$ (in particular its volume when Ω is coincident with an extrinsic ball), the behaviour in $P - \Omega$ should not be very surprising, at least as regards the number of ends.

This rigidity allows us to obtain the upper or lower bound for certain global properties (the number of ends) by studying local properties in a domain (the volume). The geometric restraints that we impose in order to obtain this local-global relation often imply an accurate behaviour of the radial mean curvature, and a decay in the second fundamental form. An inverse way to understand those rigidity results are the gap results which we obtain under the assumptions of Chapter 5 for the particular case of having a single end. The easiest statement of a gap result is for a minimal surface immersed in $\mathbb{R}^n$ with finite total curvature. In this case, if the surface has only one end, the surface is isometric to $\mathbb{R}^2$. Thus, the knowledge of the (global) number of ends also implies having knowledge about the local domains (which are all isometric to domains of $\mathbb{R}^2$).

Returning to the case of minimal surfaces in $\mathbb{R}^3$, if we study the periodic minimal surfaces we can see how the local-global relation may be broken if we do not impose any condition on the decay of the second fundamental form, as can be seen in Figure 1.4. Hence, our results might be understood as the characterisation of the geometric restrictions in an isometric immersion that imply that kind of local-global rigidity.

Finally, it must be noted that the results of this study are based on assumptions that we discussed at the beginning, i.e. a manifold N with a pole and its radial sectional curvatures bounded, an immersion of which we know its mean radial curvature $\varphi : P \to N$ from the submanifold P to the manifold N, and in conclusion we obtain results that tell us some intrinsic properties of P (topological type, number of ends and Cheeger constant). Obviously for all our results we can reverse this pattern: assuming that certain intrinsic properties of P are known, we can

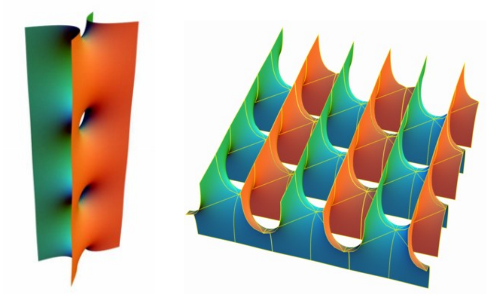


Figura 1.4: Single periodic Scherk surface (left) and the doubly periodic Scherk surface (right). They are both of finite total curvature and have no decay for the norm of the second fundamental form. Although they have very similar (local) regions, their (global) asymptotic behaviour is quite different. And although both of them have a single end, neither of them is isometric to $\mathbb{R}^2$

deduce certain properties (mainly related with their mean radial curvature) of the immersion $\varphi : P \to N$ if we suppose that N has a pole and has its radial curvature bounded appropriately. To finish, the following theorem is an example of these dual statements:

Theorem 1.6.1. *Let P^m be a complete and non-compact Riemannian manifold with zero Cheeger constant $\mathcal{I}_\infty(P)$ (i.e. $\mathcal{I}_\infty(P) = 0$). Then, there is no immersion from P to a Cartan-Hadamard manifold with strictly negative curvature.*

The proof is as follows: given a minimal immersion from P to a Cartan-Hadamard manifold of strictly negative curvature, from the results in Chapter 4 we have that the Cheeger isoperimetric constant $\mathcal{I}_\infty(P)$ of the (sub)manifold P is positive (see Theorem B in Chapter 4, where it is proved that every minimal submanifold P properly immersed in a Cartan-Hadamard manifold N of sectional curvatures $K_N \leq b < 0$ has Cheeger constant $\mathcal{I}_\infty(P) \geq (m-1)\sqrt{-b}$). Hence, there are no minimal immersions from manifolds of zero Cheeger constant to Cartan-Hadamard manifolds with strictly negative curvature.

On the other hand, Theorem B of Chapter 4 is related to the parabolicity/hyperbolicity of the submanifold in the following way:

Focusing on the fundamental tone $\lambda^*(P)$, we have that (see [Cheeger70], [Cha84], [Cha01])

$$\lambda^*(P) \geq \frac{\mathcal{I}_\infty(P)^2}{4} \tag{1.6.1}$$

Therefore, if the Cheeger isoperimetric constant of P is positive, the fundamental tone of P is positive, which implies that P is hyperbolic (non-parabolic) (see [Gri99]).

In particular, therefore, the above theorem implies that there is no minimal and proper immersion from a parabolic manifold to a Cartan-Hadamard manifold of strictly negative curvature.

1.7 Further research lines

This line of research could be continued by studying other problems of geometric analysis on submanifolds using the extrinsic distance and the comparisons with a model space. Specifically, we might consider:

1. The relation between the volume growth and the fundamental tone of a submanifold in an ambient manifold with a pole.
2. The Laplacian spectrum for the Dirichlet problem in domains of a submanifold and its relation with volume growth and the number of ends.
3. Look for the relation between the capacity of an extrinsic annulus in a submanifold of an ambient manifold with a pole and its local and global properties.
4. Generalise the Jellet theorem (see [Cha84], [Cha93]) for submanifolds properly immersed in ambient manifolds with a pole and explore its applications.
5. Study in greater depth the isoperimetric problem in submanifolds giving lower bounds for the isoperimetric profile.

2

Anàlisi amb la distància extrínseca

2.1 Introducció

L'anàlisi geomètric relaciona la geometria d'una varietat diferenciable amb l'anàlisi de les funcions definides sobre aquesta varietat. Donada una varietat N, les tècniques de l'anàlisi geomètric ens permeten relacionar les propietats geomètriques de N amb les propietats analítiques de les funcions definides sobre N. L'anàlisi geomètric en subvarietats ens permet donada una immersió $\varphi : P \to N$ de la varietat riemanniana P a la varietat riemanniana N, relacionar les propietats analítiques de les funcions definides sobre N i les propietats geomètriques de la immersió φ amb les propietats geomètriques de P. El procediment que seguim en aquest treball és el següent:

1. Partim d'una funció $f : N \to \mathbb{R}$ (o un operador diferencial de la mateixa $\mathcal{O}_f : T_pN \times T_pN \times \cdots \times T_pN \to \mathbb{R}$) del qual coneixem el seu comportament per les propietats geomètriques i topològiques de N.

2. Construïm la restricció de la funció (o del operador) a P i estudiem les seues propietats per mitjà de les propietats de la immersió .

3. Estudiem quines propietats geomètriques i topològiques podem deduir de P sabent el comportament que tenen les restriccions de les funcions i els operadors anteriors.

És a dir, partim d'una immersió $\varphi : P \to N$ de la qual coneixem la geometria i topologia de N. Amb aquesta informació deduïm algunes de les propietats de les funcions i operadors de N. Tenint present les propietats de φ deduïm les propietats de les funcions i operadors restringits a P i amb aquesta informació intentem deduir algunes propietats topològiques i geomètriques de P.

En aquest capítol introduïm principalment tots els conceptes que fan possible aquest procediment: l'estudi de les funcions i operadors sobre varietats de Riemann, la restricció d'aquestes funcions i operadors per mitjà de les immersions i, finalment, la relació entre les propietats de les funcions i operadors amb la geometria i la topologia; principalment, amb teoria de Morse i la relació entre el Hessià de la distància i les curvatures seccionals radials donada per Greene i Wu.

En tot aquest esquema nosaltres optem per la funció distància. Per tant ens cal una varietat N amb un comportament adient d'aquesta funció. Aquestes varietats són el que anomenem varietats amb pol. Ens cal estudiar també la restricció de la distància sobre la varietat P que anomenarem distància extrínseca. Així com la restricció d'alguns operadors definits amb la funció distància i la construcció d'espais models de comparació.

2.2 Operadors diferencials en varietats de Riemann

En una varietat de Riemann hi ha generalització dels operadors diferencials usuals per al càlcul vectorial en $\mathbb{R}^3$ com puguen ser el gradient, la divergència o el laplacià. Comencem primer amb la derivada direccional:

Definició 2.2.1. *(derivada direccional, v. [Cha84]) Siga M una varietat, donat $p \in M$ i una funció real f de tipus C^1 definida en un entorn de p, aleshores per a cada $X \in T_pM$ associem la derivada direccional de f en direcció X, que escriurem com $X(f)$,*

$$X(f) = (f \circ \omega)'(0), \tag{2.2.1}$$

on $\omega(t)$ és qualsevol corba tal que $\omega(0) = p$ i $\omega'(0) = X$

Es pot demostrar que de fet la definició anterior no depèn de la corba emprada. Definirem el gradient com

Definició 2.2.2. *(gradient, v. [O'N83]) El gradient ∇f d'un camp escalar f en la varietat M^n és el camp vectorial mètricament equivalent a la diferencial df. O siga:*

$$\langle \nabla f, X \rangle = df(X) = X(f). \tag{2.2.2}$$

On X és un camp vectorial definit sobre M.

Per a la divergència podem utilitzar una definició:

Definició 2.2.3. *(divergència, v. [dC92]) Siga M una varietat de Riemann, amb connexió de Levi-Civita ∇ (i.e la conexió métrica lliure de torsió). Siga X un camp vectorial. Definim la divergència de X com una funció div $X : M \to \mathbb{R}$ donada per*

$$div\ X(p) = traça\left(Y(p) \to \nabla_Y X(p)\right), \tag{2.2.3}$$

en $p \in M$.

O un altra forma equivalent de definir la divergència seria fer us del següent teorema

Teorema 2.2.4. *(Teorema de la divergència veure [Gri09]) Per qualsevol camp vectorial suau X definit sobre una varietat riemanniana M, existeix una única funció suau en M, anomenada div X, tal que satisfà la següent igualtat*

$$\int_M (div\ X)\, u d\nu = -\int_M \langle X, \nabla u \rangle d\nu \tag{2.2.4}$$

per a tota funció $u \in C_0^\infty(M)$.

Recordem que una funció és suau i de suport compacte ($f \in C_0^\infty(M)$) si és suau i

$$\text{supp } f = \overline{\{x \in M \,:\, f(x) \neq 0\}} \tag{2.2.5}$$

és compacte.

Nota 2.2.5. *Observem que aquesta darrera definició de divergència que és completament equivalent a l'anterior ens permet estendre la divergència per a un camp que no siga C^1.*

El teorema de la divergència del qual fa ús la definició darrera de divergència pot ser enunciat de forma alternativa com

Teorema 2.2.6. *(Teorema de la divergència, veure [Cha93], [Cha84]) Siga Ω una varietat amb vora $\partial\Omega$, denotarem la densitat de la mesura en Ω per $d\mu$ i la densitat de mesura en $\partial\Omega$ per $d\sigma$, denotarem amb ν el camp vectorial unitari normal cap a fora sobre $\partial\Omega$, aleshores si X es un camp vectorial que és C^1 en $\overline{\Omega}$ i té suport compacte en $\overline{\Omega}$ es satisfà que:*

$$\int_\Omega (div\ X) d\mu = \int_{\partial\Omega} \langle X, \nu \rangle d\sigma \tag{2.2.6}$$

La divergència té una interpretació clara: donat un camp vectorial X sobre la varietat riemanniana M, considerem el flux $\{\Phi_t\}$ induït pel camp X en M. Si fixem un conjunt compacte K en M, podem observar com evoluciona el seu volum per acció del flux

$$v(t) = \int_{\Phi_t(K)} d\mu, \tag{2.2.7}$$

on μ és la mesura riemanniana en M. Aleshores la divergència ens dóna (veure [Cha84])

$$v'(0) = \int_K \text{div } X\, d\mu. \tag{2.2.8}$$

Així div X mesura la distorsió infinitesimal del volum originada pel flux generat per X.

Per un altra part podem definir el hessià com:

Definició 2.2.7. *(hessià, v. [dC92]) Siga $f : M \to \mathbb{R}$ una funció diferenciable sobre la varietat M. Definim el hessià, $Hess_p(f)$ de f en el punt $p \in M$ com l'operador bilineal $Hess_p f : T_pM \times T_pM \to \mathbb{R}$ tal que*

$$Hess_p f(X, Y) := \langle \nabla_X \nabla f, Y \rangle \tag{2.2.9}$$

per a qualsevol X i qualsevol Y en T_pM.

El hessià és bilineal i simètric $Hess_p f(X, Y) = Hess_p f(Y, X)$. Una propietat important del hessià és la següent proposició

Proposició 2.2.8. *Siga $f : \mathbb{R} \to \mathbb{R}$ una funció suau definida sobre $\mathbb{R}$ i $g : M \to \mathbb{R}$ una funció suau definida sobre la varietat riemanniana i siguen X i Y dos vectors qualsevols pertanyents a T_pM aleshores*

$$\mathrm{Hess}_p\, f \circ g(X,Y) = (f'' \circ g)(p)\,\langle \nabla g, X\rangle\langle \nabla g, Y\rangle + (f' \circ g)(p)\;\mathrm{Hess}\, g(X,Y) \tag{2.2.10}$$

Demostració.

$$\begin{aligned} \mathrm{Hess}_p\, f \circ g(X,Y) &= \langle \nabla_X \nabla(f\circ g), Y\rangle = \langle \nabla_X \big((f'\circ g)\nabla g\big), Y\rangle \\ &= X((f'\circ g))\langle \nabla g, Y\rangle + (f'\circ g)(p)\,\mathrm{Hess}\, g(X,Y) = \\ &= (f''\circ g)(p)\langle \nabla g, X\rangle\langle \nabla g, Y\rangle + (f'\circ g)(p)\,\mathrm{Hess}\, g(X,Y) \end{aligned} \tag{2.2.11}$$

□

Definició 2.2.9. *(laplacià)[O'N83] El laplacià d'una funció és la divergència del gradient:*

$$\Delta f = div\ (\nabla f) \tag{2.2.12}$$

Nota 2.2.10. *També haguérem pogut definir el laplacià com la traça del hessià definit anteriorment.*

Propietats del gradient, la divergència i el laplacià

Seguint [Cha84] podem enumerar les següents propietats per al gradient, la divergència i el laplacià

Proposició 2.2.11. *([Cha84], [Gri09]) Siga M una varietat de Riemann, siguen f i g dues funcions suaus i siguen X, Y dos camps vectorials suaus definits sobre M, aleshores*

1. $\nabla(f+h) = \nabla f + \nabla h$
2. $\nabla(fh) = h\nabla f + f\nabla h$
3. $div\ (X+Y) = div\ X + div\ Y$
4. $div\ (fX) = f\,div\ X + \langle \nabla f, X\rangle$
5. $\Delta(f+h) = \Delta f + \Delta h$
6. $\Delta(fh) = h\Delta f + 2\langle \nabla f, \nabla h\rangle + f\Delta h$
7. $div\ (h\nabla f) = h\Delta f + \langle \nabla h, \nabla f\rangle$

2.3 Operadors diferencials i immersions

Immersions i immersions pròpies

Per fixar conceptes recordem el concepte d'immersió entre dues varietats riemannianes

Definició 2.3.1. *([dC92]) Siguen (P^m, g_P) i (N^n, g_N) dues varietats de Riemann. A una aplicació diferenciable $\varphi : M \to N$ l'anomenarem* immersió *si $d\varphi_p : T_pP \to T_{\varphi(p)}N$ és injectiva per a tot punt $p \in P$. A més a més si $\varphi^* g_N = g_P$ direm que la immersió és una* immersió isomètrica.

Al llarg d'aquesta tesi direm que P és una subvarietat immersa en N si existeix una immersió isomètrica $\varphi : P \to N$.

Com $\varphi : P^m \to N^n$ és una immersió, aleshores per a cada $p \in P$, existeix un entorn $U \subset P$ de p tal que $\varphi(U) \subset N$ és una subvarietat de N (açò significa que existeix un entorn $V \subset N$ de $\varphi(p)$ i un difeomorfisme $f : V \to W \subset \mathbb{R}^n$ tal que f porta $\varphi(U) \cap V$ difeomorficament a un subconjunt obert d'un subespai de $\mathbb{R}^m \subset \mathbb{R}^n$) (veure [dC92]).

En aquest treball parlarem principalment d'immersions pròpies. Recordem que

Definició 2.3.2. *Una aplicació entre dos espais topològics $f : X \to Y$ és pròpia si la preimatge de cada conjunt compacte en Y és compacte en X.*

Així el concepte d'immersió pròpia senzillament serà

Definició 2.3.3. *Siguen P^m i N^n dues varietats de Riemann. A una aplicació diferenciable $\varphi : M \to N$ l'anomenarem immersió isomètrica pròpia si φ és una immersió isomètrica i φ és una aplicació pròpia entre P i N.*

En cas que la immersió siga entre dues varietats completes resulta senzill caracteritzar les immersions pròpies per la següent proposició, que pot ser entesa en el sentit de que $\varphi : P \to N$ és una immersió isomètrica propia entre dues varietats completes si, i sols si, $\rho^N \to \infty$ quan $\rho^P \to \infty$.

Proposició 2.3.4. *La immersió isomètrica $\varphi : P \to N$ entre la varietat completa P i la varietat completa N serà pròpia si, i sols si, per a qualsevol subconjunt $C \subset P$ tal que $\sup_{x\in C} \rho^P(x) = \infty$ tenim que $\sup_{y\in\varphi(C)} \rho^N(y) = \infty$ on ρ^P i ρ^N són la funció distància en P i N respectivament.*

Demostració. Anem a demostrar primer que si la immersió és pròpia aleshores per a qualsevol $C \subset P$ tal que $\sup_{x\in C} \rho^P(x) = \infty$ tenim que $\sup_{y\in\varphi(C)} \rho^N(y) = \infty$.

Suposem que existeix un $C_0 \subset P$ tal que $\sup_{x\in C_0} \rho^P(x) = \infty$ però $\sup_{y\in\varphi(C_0)} \rho^N(y) < \infty$ i suposem, a més a més, que la immersió és pròpia i demostrem que aquestes dues coses juntes són contradictòries.

En una varietat completa, pel teorema de Hopf-Rinow (veure [dC92]), que un conjunt siga compacte és equivalent a dir que és tancat i està fitat. Partim per tant d'aquest $C_0 \subset P$ tal que $\sup_{x\in C_0} \rho^P(x) = \infty$, com no està fitat, evidentment C_0 no és compacte, però com $\sup_{y\in\varphi(C_0)} \rho^N(y) < \infty$, $\varphi(C_0)$ està fitat en N i el mateix passa amb la seua clausura $\overline{\varphi(C_0)}$ que és per tant compacta en N. Així com hem suposat que la immersió és pròpia $\varphi^{-1}(\overline{\varphi(C_0)})$ és compacte en P i per tant fitat en P el que implica que

$$\sup_{x\in\varphi^{-1}(\overline{\varphi(C_0)})} \rho^P(x) < \infty. \tag{2.3.1}$$

Però com $C_0 \subset \varphi^{-1}(\varphi(C_0)) \subset \varphi^{-1}(\overline{\varphi(C_0)})$ aleshores

$$\sup_{x \in C_0} \rho^P(x) < \infty, \tag{2.3.2}$$

en clara contradicció amb la suposició de que $\sup_{x \in C_0} \rho^P(x) = \infty$.

Ara cal demostrar-ho en sentit contrari, és a dir, si es compleix que per a qualsevol $C \subset P$ amb $\sup_{x \in C} \rho^P(x) = \infty$ tenim que $\sup_{y \in \varphi(C)} \rho^N(y) = \infty$ aleshores la immersió és pròpia. Aquest enunciat és equivalent a afirmar que si la immersió no és pròpia existeix almenys un subconjunt $C_1 \subset P$ que compleix $\sup_{x \in C_1} \rho^P(x) = \infty$ però que $\sup_{y \in \varphi(C_1)} \rho^N(y) < \infty$.

Com la immersió no és pròpia existeix K_0 compacte en N (fitat i tancat) tal que $\varphi^{-1}(K_0)$ no és compacte en P. Com K_0 és compacte i φ és continua aleshores $\varphi^{-1}(K_0)$ és tancat, com sabem que $\varphi^{-1}(K_0)$ no és compacte en P i P és completa, la única opció és que $\varphi^{-1}(K_0)$ no estiga fitat, per tant

$$\sup_{x \in \varphi^{-1}(K_0)} \rho^P(x) = \infty. \tag{2.3.3}$$

Però com $\varphi(\varphi^{-1}(K_0)) = K_0 \cap \varphi(P) \subset K_0$

$$\sup_{y \in \varphi(\varphi^{-1}(K_0))} \rho^N(y) = \sup_{y \in \varphi(P) \cap K_0} \rho^N(y) \leq \sup_{y \in K_0} \rho^N(y) < \infty. \tag{2.3.4}$$

Per tant $C_1 = \varphi^{-1}(K_0)$ compleix les condicions que volíem. □

Nota 2.3.5. *Cal fer menció que en la proposició anterior no hem especificat cap punt per a que ens servira com origen per mesurar les distàncies ni en P ni N. De fet el resultat implica tan sols el comportament asimptòtic de la distància .*

Segona forma fonamental d'una immersió isomètrica

Per poder estendre els operadors diferencials sobre una immersió $\varphi : P \to N$, cal que analitzem prèviament els espais tangents a les varietats P i N.

Per simplificar la notació identificarem cada vector $v \in T_pP$ per a $p \in P$ amb $d\varphi_p(v) \in T_{\varphi(p)}N$ i de forma local p amb $\varphi(p)$. És fàcil adonar-se de la descomposició dels espais vectorials:

$$T_pN = T_pP \oplus (T_pP)^\perp \tag{2.3.5}$$

On $(T_pP)^\perp$ és el complement ortogonal de T_pP en T_pN.

Per a qualsevol vector $v \in T_pN$, podem escriure:

$$v = v^T + v^\perp, \quad v^T \in T_pP, \quad v^\perp \in (T_pP)^\perp \tag{2.3.6}$$

Aleshores tenim definits dos fibrats vectorials sobre P: el fibrat tangent TP, amb fibra T_pP sobre $p \in P$, i el fibrat normal νP amb fibra $(T_pP)^\perp$ sobre $p \in P$. Amb l'ajuda de la connexió de Levi-Civita en la varietat ambient ∇^N, podem definir dues connexions sobre aquests fibrats vectorials:

1. La connexió tangent o induïda $D^T : \mathcal{X}(P) \times \mathcal{X}(P) \to \mathcal{X}(P)$, donada per la component tangent
$$D^T_X Y = (\nabla^N_X Y)^T \tag{2.3.7}$$
amb $X, Y \in \mathcal{X}(P)$.

2. I, la connexió normal $D^\perp : \mathcal{X}(P) \times \mathcal{X}^\perp(P) \to \mathcal{X}^\perp(P)$, donada per la component normal
$$D^\perp_X V = (\nabla^N_X V)^\perp \tag{2.3.8}$$
amb $X \in \mathcal{X}(P)$ i $V \in \mathcal{X}^\perp(P)$.

Definim la segona forma fonamental de P, $B^P : \mathcal{X}(P) \times \mathcal{X}(P) \to \mathcal{X}^\perp(P)$ com
$$B^P(X, Y) := \nabla^N_X Y - D^T_X Y \tag{2.3.9}$$
per a $X, Y \in \mathcal{X}(P)$.

I definim l'aplicació de Weingarten $L : \mathcal{X}^\perp(P) \times \mathcal{X}(P) \to \mathcal{X}(P)$ com
$$L_V X := D^\perp_X V - \nabla^N_X V \tag{2.3.10}$$
per a $V \in \mathcal{X}^\perp(P)$ i $X \in \mathcal{X}(P)$.

Per resumir tots els resultats clàssics sobre aquestes dues connexions, sobre la segona forma fonamental i l'aplicació de Weingarten, enunciem la següent proposició

Proposició 2.3.6. *(Veure [dC92], [Cha93],[KN96],[Che84]) Siga $\varphi : P^m \to N^n$ una immersió isomètrica de la varietat $m-$dimensional P en la varietat $n-$dimensional N. Denotem per D^T i per $D^\perp$ les connexions sobre el fibrat tangent TP i sobre el fibrat normal νP respectivament. Aleshores*

1. *La connexió $D^\perp$ és una connexió mètrica sobre el fibrat normal νP.*

2. *La connexió D^T és una connexió mètrica i lliure de torsió sobre el fibrat tangent TP (i.e la connexió D^T és la connexió de Levi-Civita ∇^P).*

3. *La segona forma fonamental B^P és bilineal i simètrica.*
$$B^P(X, Y) = B^P(Y, X)$$

4. *L'aplicació de Weingarten és una transformació lineal autoadjunta*
$$\langle L_V X, Y\rangle = \langle X, L_V Y\rangle$$

5. *La segona forma fonamental i l'aplicació de Weingarten estan relacionades per*
$$\langle L_V X, Y\rangle = \langle B^P(X, Y), V\rangle.$$

6. *La subvarietat P és totalment geodèsica en N (les corbes geodèsiques de P són geodèsiques en N) si, i sols si, $B^P = 0$.*

Recordem també la definició de vector curvatura mitja, de subvarietat minimal i de subvarietat totalment umbilical

Definició 2.3.7. *Siga $\varphi : P^m \to N^n$ una immersió isomètrica de la varietat $m-$dimensional P en la varietat $n-$dimensional N. Al camp vectorial H de νP donat per*

$$H = \frac{1}{m} tr\ (B^P). \tag{2.3.11}$$

L'anomenarem vector curvatura mitja. *És a dir, si $\{E_1, \cdots .E_m\}$ forma una base ortonormal en T_pP aleshores el vector curvatura mitja en $p \in P$ és*

$$H = \frac{1}{m} \sum_{i=1}^{m} B^P(E_i, E_i). \tag{2.3.12}$$

Direm que la subvarietat P és minimal si

$$H = 0. \tag{2.3.13}$$

Direm que la subvarietat P és totalment umbilical si

$$B^P(X, Y) = \langle X, Y \rangle H. \tag{2.3.14}$$

Nota 2.3.8. *De la propietat 5 de la proposició anterior podem deduir que donada una base base ortonormal $\{e_i\}_{i=1}^{n-m}$ de $(T_pP)^\perp$,*

$$H = \frac{1}{m} \sum_{k=1}^{n-m} traça\ (L_{e_k}) e_k. \tag{2.3.15}$$

Gradient i hessià sobre immersions

En aquesta secció descriurem com actuen els operadors diferencials sobre les funcions $f : P \to \mathbb{R}$ restringides a les subvarietats. Com les subvarietats són també varietats riemannianes podem definir el gradient, la divergència, el hessià i el laplacià de forma intrínseca sobre les subvarietats. Aquests darrers operadors els denotarem amb un superíndex P (per exemple: ∇^P) per poder distingir-los dels operadors sobre les varietats ambient que denotarem amb un superíndex N (per exemple: ∇^N).

Considerem ara la funció suau $f : N^n \to \mathbb{R}$ definida sobre la varietat ambient N, definirem com la restricció $f|_P$ de la funció f sobre la subvarietat P a la composició: $f|_P = f \circ \varphi : P^m \to \mathbb{R}$.

Donat qualsevol punt q en P i qualsevol vector X en T_qP, si fem la identificació $X = d\varphi(X)$ podem escriure:

$$\langle \nabla^P f|_P, X \rangle = df|_P(X) = df(X) = \langle \nabla^N f, X \rangle \tag{2.3.16}$$

Per tant podem escriure:

$$\nabla^N f = \nabla^P f|_P + \left(\nabla^P f|_P\right)^\perp \tag{2.3.17}$$

On $\left(\nabla^P f|_P\right)^\perp$ és perpendicular a T_qP en q $(\left(\nabla^P f|_P\right)^\perp \in (T_pP)^\perp)$.

De forma semblant per al hessià podem obtenir la següent proposició

Proposició 2.3.9. *Siga la immersió isomètrica $\varphi : P^m \to N^n$, i siga $f : N \to \mathbb{R}$ una funció suau, denotem per $f|_P : P \to \mathbb{R}$ la seua restricció a P aleshores:*

$$\operatorname{Hess}^P f|_P(X,Y) = \operatorname{Hess}^N f(X,Y) + \langle B^P(X,Y), \nabla^N f\rangle \tag{2.3.18}$$

Demostració. Immediat de la descomposició del gradient donada anteriorment i de la fórmula de Gauss

$$\begin{aligned}\operatorname{Hess}^P f|_P(X,Y) &= \langle \nabla^P_X \nabla^P f|_P, Y\rangle = \langle \nabla^N_X \nabla^P f|_P, Y\rangle = \langle \nabla^N_X \left(\nabla^N f - (\nabla^P f|_P)^\perp\right), Y\rangle \\ &= \operatorname{Hess}^N f(X,Y) - \langle \nabla^N_X (\nabla^P f|_P)^\perp, Y\rangle \\ &= \operatorname{Hess}^N f(X,Y) + \langle (\nabla^P f|_P)^\perp, \nabla^N_X Y\rangle \\ &= \operatorname{Hess}^N f(X,Y) + \langle (\nabla^P f|_P)^\perp, B^P(X,Y)\rangle \\ &= \operatorname{Hess}^N f(X,Y) + \langle \nabla^N f, B^P(X,Y)\rangle \end{aligned} \tag{2.3.19}$$

□

Aplicant ara la proposició 2.2.8 arribem immediatament a un important corol·lari:

Corol·lari 2.3.10. *Siga la immersió isomètrica $\varphi : P^m \to N^n$, siguen $g : N \to \mathbb{R}$ i $f : \mathbb{R} \to \mathbb{R}$ funcions suaus, i siguen X i Y dos vectors pertanyents a T_pP, aleshores:*

$$\begin{aligned}\operatorname{Hess}^P f \circ g|_P(X,Y)|_p &= (f'' \circ g)(p)\langle \nabla^P g|_P, X\rangle\langle \nabla^P g|_P, Y\rangle \\ &\quad + (f' \circ g)(p)\left(\operatorname{Hess}^N g(X,Y) + \langle \nabla g, B^P(X,Y)\rangle\right)\end{aligned} \tag{2.3.20}$$

2.4 Funcions, punts crítics, teoria de Morse, teorema de Sard i formula de la coàrea

En aquesta secció introduirem unes breus nocions sobre teoria de Morse que ens seran útils més endavant. Siga T_pM l'espai tangent a la varietat diferenciable M en el punt $p \in M$, siga $f : M \to N$ una aplicació C^∞ entre la varietat M i la varietat N amb $f(p) = q$, i siga $f_* : T_pM \to T_qN$ l'aplicació lineal induïda. Aleshores definim el punt crític com:

Definició 2.4.1. *Siga $f : M \to \mathbb{R}$ una funció suau definida sobre la varietat diferenciable M. Un punt $p \in M$ és anomenat un punt crític de f si l'aplicació induïda $f_* : T_pM \to T_{f(p)}\mathbb{R}$ és zero, és a dir $f_* = 0$ en p.*

Nota 2.4.2. *Cal fer notar que si en la definició anterior M és una varietat de Riemann, $p \in M$ és un punt crític de f si i sols si , p és un punt on s'anul·la ∇f ($\nabla f = 0$ en p) (veure [dC94]).*

Podem definir ara els valors regulars i els valors crítics

Definició 2.4.3. *Siga $f : M \to N$ una aplicació C^∞, al punt $q \in N$ serà anomenat valor crític de f si $f^{-1}(q)$ conté algun punt crític de f. En cas contrari direm que q és un valor regular de f.*

Es pot demostrar que el conjunt de valors crítics forma un conjunt de mesura nul·la utilitzant l'important teorema de Sard:

Teorema 2.4.4. *(Teorema de Sard, veure [GG73],[Spi79],[Che84]) Siga* $f : M^n \to N^m$ *una aplicació* C^∞ *entre dues varietats, aleshores el conjunt de valors crítics de* f *formen un conjunt de mesura de Lebesgue nul·la en* N.

Per a un punt crític $p \in M$ de f podem definir la forma bilineal simètrica $D^2 f(p)(u, v) := X(Y(f))(p)$, on X, Y són camps vectorials sobre M amb $X_p = u$, $Y_p = v$. Un punt crític $p \in M$ de la funció f s'anomena *no degenerat* si $D^2 f(p)$ és no degenerat, i.e, l'únic vector $u \in T_pM$ tal que $D^2 f(u, v) = 0$ per a qualsevol $v \in T_pM$ és el vector nul ($u = 0$).

Nota 2.4.5. *Observem com les definicions anteriors de punt crític, valor regular, etc, no fan ús de l'estructura riemanniana de la varietat, tan sols de l'estructura diferencial de la varietat. Fent un càlcul senzill (seguint [Sak96]) podem demostrar que sobre un punt crític* $p \in M$ *de la funció* f *definida sobre la varietat riemanniana*

$$\begin{aligned} \operatorname{Hess}_p f(X, Y) &= \langle \nabla_X \nabla f, Y \rangle = (X(Y(f)))(p) - \langle \nabla f, \nabla_X Y \rangle_p \\ &= (X(Y(f)))(p) = D^2 f(p)(X, Y) \end{aligned} \tag{2.4.1}$$

Així, direm que sobre una varietat riemanniana M *el punt crític* $p \in M$ *de la funció* f *és no degenerat si el hessià* $\operatorname{Hess}_p f$ *és no degenerat.*

Una funció C^∞ que admet únicament punts crítics no-degenerats és anomenada una *funció de Morse*. És conegut (veure [Sak96]) que qualsevol funció C^∞ sobre M pot ser aproximada per funcions de Morse (respecte a la topologia C^∞).

Definim ara els següents conjunts de nivell

Definició 2.4.6. *Siga* M *una varietat diferenciable, i siga* $f : M \to \mathbb{R}$ *una funció suau definida sobre* M. *per a cada* $a \in \mathbb{R}$ *denotarem*

$$\begin{aligned} M^a &:= \{x \in M \;:\; f(x) \leq a\} \\ \partial M^a &:= \{x \in M \;:\; f(x) = a\} \end{aligned}$$

Ara donarem un resultat capital per aquest treball:

Teorema 2.4.7. *(Veure [Sak96], [Mil63])Suposem que* $f^{-1}([a, b])$ *és compacte y no conté punts crítics de* f. *Aleshores* $f^{-1}([a, b])$ *és difeomorf a* $f^{-1}(a) \times [a, b]$, *i* M^a *és difeomorf a* M^b. *A més a més, l'aplicació d'inclusió* $i : M^a \hookrightarrow M^b$ *dóna una equivalència d'homotopia. De fet el difeomorfisme ve donat pel flux del camp vectorial* $\frac{\nabla f}{\|\nabla f\|^2}$ *on* ∇f *és el vector gradient de* f.

Si a és un valor regular de f, pel teorema de la funció inversa (veure [War83],[Mil63] o [Sak96]) M^a és una varietat suau amb vora i la vora ($f^{-1}(a)$) és una subvarietat suau de M.

Si a més a més la funció f és pròpia tindrem que

Proposició 2.4.8. *Siga $f : M^n \to \mathbb{R}$ una funció pròpia C^∞ definida sobre una varietat de Riemann M^n. Per a cada t pertanyent al conjunt de valors regulars de f, $f^{-1}(t)$ és una hipersuperfície compacta de M, i el vector gradient $\nabla f(q)$, és perpendicular a $f^{-1}(t)$.*

Continuem ara amb una varietat riemanniana (M, g) i una funció $f : M \to \mathbb{R}$ de tipus C^∞ sobre M. Si c és un valor regular de f existeix un interval obert (a, b) contenint a c tal que el camp $X = \frac{\nabla f}{\|\nabla f\|^2}$ està ben definit en l'obert $f^{-1}(a, b)$ de M.

Siga $q \in f^{-1}(c)$ (és a dir $q \in \partial M^c$), denotem per φ_t el flux generat per X (això és, $\frac{d}{dt}\varphi_t(p) = X_p$ i $\varphi_0(p) = p$). Aleshores:

$$\frac{d}{dt}f(\varphi_t(q)) = \langle \nabla f, X\rangle_{\varphi_t(q)} = \langle \nabla f, \frac{\nabla f}{\|\nabla f\|^2}\rangle_{\varphi_t(q)} = 1. \tag{2.4.2}$$

Com $f(\varphi_0(q)) = f(q) = c$, integrant l'equació anterior arribem a

$$f(\varphi_{t-c}(q)) = t. \tag{2.4.3}$$

Definim ara el difeomorfisme (local) $\Phi : \partial M^c \times (a, b) \to f^{-1}(a, b)$ donat per

$$\Phi(q, t) = \varphi_{t-c}(q), \tag{2.4.4}$$

per a tot $q \in \partial M^c$ i tot $t \in (a, b)$.

Aleshores, Φ satisfà:

$$\begin{cases} f(\Phi(q, t)) = t \\ \|\frac{\partial}{\partial t}\Phi(q, t)\|^2 = \frac{1}{\|\nabla f\|^2}. \end{cases} \tag{2.4.5}$$

Sent $\frac{\partial}{\partial t}\Phi(q, t)$ perpendicular a ∂M^t per a tot $t \in (a, b)$.

Per a tot $p \in f^{-1}(a, b)$ existeixen uns únics $(q, t) \in \partial M^c \times (a, b)$ tals que $q = \Phi(q, t)$. Podem escollir així, per a l'espai tangent T_pM la següent base ortogonal $\{\frac{\partial}{\partial t}, e_2, \cdots, e_n\}$, on $\frac{\partial}{\partial t}$ denota el vector tangent a la corba $\Phi(q, t)$ en el punt p (i.e, $\frac{\partial}{\partial t}\Phi(q, t)$), i $\{e_2, \cdots, e_n\}$ formen una base ortonormal de ∂M^t. Si denotem per $\{w^2, \cdots, w^n\}$ les formes duals a $\{e_2, \cdots, e_n\}$ (i.e, $w^i(e_j) = \delta_i^j$) , obtenim per a la densitat de mesura riemaniana $d\mu_g$ associada a la mètrica g:

$$d\mu_g = \sqrt{\det g}\, dt \wedge w^2 \wedge \cdots \wedge w^n. \tag{2.4.6}$$

Com la mètrica g és diagonal en la base escollida

$$\det g = \langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t}\rangle \Pi_{i=2}^n \langle e_i, e_i\rangle = \|\frac{\partial}{\partial t}\Phi(q, t)\|^2 \Pi_{i=2}^n \langle e_i, e_i\rangle = \frac{1}{\|\nabla f\|^2} \quad , \tag{2.4.7}$$

per tant

$$d\mu_g = \frac{1}{\|\nabla f\|}\, dt \wedge w^2 \wedge \cdots \wedge w^n. \tag{2.4.8}$$

Com per una altra part ∂M^t és una hipersuperfície de M amb vector unitari normal $\frac{\nabla f}{\|\nabla f\|}$ i amb mètrica g_t induïda per la inclusió $i : \partial M^t \to M$ (és a dir $g_t = i^*g$) és senzill adonar-se que $d\mu_{g_t} = w^2 \wedge \cdots \wedge w^n$ i així

$$d\mu_g = \frac{1}{\|\nabla f\|} dt \wedge d\mu_{g_t} \tag{2.4.9}$$

Aplicant el teorema de Fubini arribem a la fórmula de la coàrea

Teorema 2.4.9. *(fórmula de la coàrea)([Sak96], [Cha84]) Amb les definicions fetes anteriorment, per a qualsevol funció integrable u en la varietat riemanniana (M, g) tenim el següent:*

- *Siga g_t la mètrica induïda en ∂M^t de g. Aleshores:*
$$\int_M u\| \nabla f\| d\mu_g = \int_{-\infty}^{\infty} dt \int_{\partial M^t} u d\mu_{g_t}. \tag{2.4.10}$$
- *$t \to \mathrm{Vol}(M^t)$ és de classe C^∞ per a un valor regular t de f tal que $\mathrm{Vol}(M^t) < \infty$, i :*
$$\frac{d}{dt}\mathrm{Vol}(M^t) = \int_{\partial M^t} \| \nabla f\|^{-1} d\mu_{g_t}. \tag{2.4.11}$$

2.5 Varietats amb pol

En aquest treball utilitzarem la funció distància per analitzar les propietats de les varietats. Anem a recordar breument algunes definicions bàsiques relacionades amb la distància en varietats riemannianes. La distància entre dos punts és

Definició 2.5.1. *(veure per exemple [dC92]) Donada una varietat riemanniana connexa la distància entre dos punts és el ínfim de les longituds de totes les corbes que uneixen aquests dos punts.*

La relació entre la distància i les geodèsiques és molt estreta. Siga M una varietat riemanniana, donada $\gamma : [0, \infty) \to M$ una geodèsica parametritzada per la longitud d'arc sabem que per a valors suficientment petits de t, $d(\gamma(0), \gamma(t)) = t$.

Per tal de poder treballar amb la distància de forma còmoda podríem centrar-nos en varietats completes

Definició 2.5.2. *(Veure per exemple [dC92]) Una varietat M és (geodèsicament) completa si per qualsevol $p \in M$, l'aplicació exponencial, exp_p, està definida per a tot $v \in T_pM$, això és, qualsevol geodèsica $\gamma(t)$ partint de p està definida per a tots els valors del paràmetre $t \in \mathbb{R}$.*

L'avantatge d'utilitzar varietats completes és que utilitzant el teorema de Hopf-Rinow ([dC92]) podem obtenir que per qualsevol $q \in M$ existeix una geodèsica γ que uneix p amb q amb $l(\gamma) = d(p, q)$ sent $l(\gamma)$ la longitud del segment de geodèsica, i que les boles geodèsiques centrades en qualsevol punt ($B_R^M(o) := \{p \in M \,:\, d(o, p) < R\}$) són conjunts connexos i precompactes.

A més a més donada una geodèsica $\gamma : I \to M$, si $\gamma([0, t_1])$ no és minimitzant (no realitza la distància) el mateix passa per a tot $t > t_1$. Per continuïtat el conjunt dels nombres $t > 0$ per als que $d(\gamma(0), \gamma(t)) = t$ és de la forma $[0, t_0]$ o de la forma $[0, \infty)$. Açò ens permet definir

Definició 2.5.3. *(Veure [dC92] i [Cha93]) Siga* $\gamma : I \to M$ *una geodèsica en la varietat rieamanniana* M*, amb* $p = \gamma(0)$*. Anomenarem* punt mínim de p al llarg de γ *al primer punt* $\gamma(t_0)$ *per al qual* γ *deixa de ser minimitzant. En cas que* $\gamma(t)$ *siga minimitzant per a tot* $t \in [0, \infty)$ *direm que no existeix punt mínim.*

Amb els punts mínims podem definir el *cut locus* i el *radi d'injectivitat*

Definició 2.5.4. *(veure [dC92] i [Cha93]) Siga* M *una varietat riemanianan, siga* $p \in M$ *un punt de* M*, anomenarem* cut locus *de* p*,* $C_m(p)$*, al lloc geomètric dels punts mínims de* p *(o siga, la unió dels punts mínims de* p *al llarg de totes les geodèsiques que parteixen de* p*).*

Anomenarem radi d'injectivitat $r_{inj}(p)$ *del punt* p *a*

$$r_{inj}(p) := d(p, C_m(p)).$$

I anomenarem radi d'injectivitat $r_{inj}(M)$ *de* M *a*

$$r_{inj}(M) = \inf_{p \in M} \{r_{inj}(p)\}$$

Podem resumir el comportament de la distància en una varietat completa amb el següent teorema

Teorema 2.5.5. *(Veure [Cha93], [dC92], [Sak96], [Pet98]) Siga* M *una varietat completa, siga* $o \in M$*, denotem com* $d_o(p) =$ *distància des de* o *a* p*. Aleshores*

1. *Donat qualsevol* $p \in M$ *existeix una geodèsica normalitzada* $\gamma : I \to M$ *que parteix de* o *(*$\gamma(0) = o$*) tal que* $\gamma(d_o(p)) = p$*.*

2. *Si* $p \in M - C_m(o)$ *existeix una única geodèsica minimitzant que uneix* o *i* p*.*

3. *La funció* $d_o : M \to \mathbb{R}^+$ *és* C^∞ *en tot* $M - \{\{o\} \cup C_m(o)\}$*.*

4. $\|\nabla d_o\| = 1$ *en tot* $M - \{\{o\} \cup C_m(o)\}$*.*

5. *Les boles geodèsiques (*$B_R^M(o) := \{p \in M \,:\, d_o(p) < R\}$*) són conjunts connexos i precompactes.*

6. *L'aplicació* $exp_o : T_oM \to B_R^M(o)$ *és injectiva per a tot valor de* $R < r_{inj}(o)$*, de fet (veure [Cha93])* $exp_p(t\,v)$ *és un diffeomorfisme en la seua imatge per a* $t \in (o, r_{inj}(p))$ *i* $v \in S_pM$ *(*$v \in T_pM$ *i* $\|v\| = 1$*).*

7. $M - C_m(o)$ *és homeomorf a una bola oberta en l'espai euclidià.*

8. *Les esferes geodèsiques (*$S_R^M(o) := \{p \in M \,:\, d_o(p) = R\}$*) són subvarietats diferenciables per tot valor de* $R < r_{inj}(o)$*.*

Les varietats completes tenen propietats que les fan molt interessants, però tenen el problema que la funció distància no és diferenciable en tots els llocs. Com podem veure en l'exemple de la figura 2.1 les esferes geodèsiques són subvarietats suaus per a tot radi menor que el radi d'injectivitat. De fet, en l'exemple les esferes geodèsiques

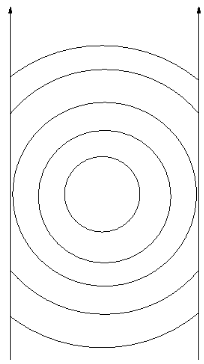


Figura 2.1: Distintes esferes geodèsiques sobre un cilindre obtingut en identificar els punts de la fletxa de la dreta amb els punts de la fletxa de l'esquerra. El cut locus per a l'origen seria la fletxa de la dreta (o l'esquerra).

són corbes suaus (sense angles) per a tot radi menor que el radi d'injectivitat però quan les esferes geodèsiques coincideixen en el cut locus de l'origen apareixen angles que fan que les nostres corbes no tinguen un vector tangent definit en aquests punts.

En aquest treball ens centrem en varietats ambient amb un pol en el sentit que es fa en el treball de Greene-Wu [GW79]

Definició 2.5.6. *([GW79]) Un punt p d'una varietat riemanniana M és un pol si, i sols si, l'aplicació exponencial $exp\ : T_pM \to M$ és un difeomorfisme.*

Nota 2.5.7. *Tant en [dC92] com en [Sak96] es defineix pol com un punt p d'una varietat riemanniana completa M que posseeix la propietat de no tenir punts conjugats (per exemple tot punt en una varietat de curvatura no positiva). Si la varietat és simplement connexa, açò implica que l'aplicació exponencial és un difeomorfisme i per tant totes dues definicions de pol són coincidents en aquest cas particular.*

Així resulta evident que tractant amb varietats amb pol no tenim perquè anar amb compte amb el cut locus (de fet les varietats amb pol no tenen cut locus) i a més a més (veure [dC92]) les varietats amb un pol són completes i difeomorfes a l'espai euclidià.

Els exemples més usuals de varietat amb pol són l'espai euclidià $\mathbb{R}^n$ i l'espai hiperbòlic $\mathbb{H}^n$, que descriurem més endavant com espais models o espais rotacionalment simètric.

Per un altra part, recordem que

Definició 2.5.8. *(Veure [dC92]) Direm que una varietat de Riemann M és una varietat de Cartan-Hadamard si és completa, simplement connexa i totes les curvatures seccionals en tots els punts de la varietat són no positius.*

Totes les varietats de Cartan-Hadamard són varietats amb pol pel teorema de Hadamard (veure [Sak96]).

Però les varietats amb pol poden tindre també curvatura positiva, en dues dimensions per exemple, el paraboloide de revolució és efectivament una varietat amb pol.

Donada una varietat amb pol (M, o), podem definir el *camp vectorial radial* (veure [GW79]) com el camp vectorial unitari ∂ definit en $M \setminus \{o\}$ tal que per qualsevol $x \in M \setminus \{o\}$, $\partial(x)$ és el vector unitari tangent a la única geodèsica que uneix al punt o i a p des de o a p.

Amb el camp radial podem definir el pla radial

Definició 2.5.9. *(Veure [GW79]) Siga (M, o) una varietat amb pol. Donat $\Pi \subset T_pM$ un subespai bidimensional de T_pM, Π serà un pla radial si, i sols si, conté a $\partial(p)$.*

I amb el pla radial podem definir la seua curvatura seccional radial

Definició 2.5.10. *(Curvatura seccional radial) (veure [GW79])) Anomenarem curvatura seccional radial d'una varietat amb un pol a la restricció de la funció curvatura seccional als plans radials.*

2.6 Distància extrínsica i boles extrínseques

Considerem ara una immersió isomètrica $\varphi : P \to (N, o_N)$, sent N una varietat amb pol o_N. Per les explicacions anteriors, queda clar que la distància riemanniana intrínseca en P no és necessàriament diferenciable. Açò fa que no puguem utilitzar importants ferramentes com la teoria de Morse, o fins i tot el teorema de Sard. En el cas bidimensional, si intentarem per exemple aplicar la fórmula de Gauss-Bonnet a les boles geodèsiques, la seua frontera com hem vist anteriorment (veure l'exemple de la figura 2.1) podria presentar angles i ens hauríem d'enfrontar en el problema de ser capaços d'estimar el seu valor.

Per poder estudiar la varietat P ens agradaria una funció C^∞ definida sobre ella. Com sabem que la distància és una funció C^∞ en $N - o_N$ podem construir una funció $r_{o_N} : P \to \mathbb{R}^+$ que siga C^∞ sobre $P - \{\varphi^{-1}(o_N)\}$ que anomenarem *distància extrínseca* i no serà altra cosa que la restricció de la distància en N a la varietat P ($r_{o_N} := d_{o_N} \circ \varphi$). O siga donat qualsevol punt $p \in P$

$$r_{o_N}(p) = \mathrm{dist}^N(o_N, \varphi(p)) \tag{2.6.1}$$

Per distingir les dues definicions de funció distància, a la distància usual sobre P l'anomenarem *distància intrínseca*. Amb la funció distància extrínseca podem definir les *boles extrínseques* D_R de radi R

$$\begin{aligned} D_R :=& \{p \in P \,:\, r_{o_N}(p) < R\} \\ =& \{p \in P \,:\, \mathrm{dist}^N(o_n, \varphi(p)) < R\} \\ =& \varphi^{-1}\left(\varphi(P) \cap B^N_R(o_N)\right) = \varphi^{-1}\left(B^N_R(o_N)\right). \end{aligned} \tag{2.6.2}$$

I la frontera d'aquestes boles extrínseques que anomenarem *esfera extrínseca* de radi R i denotarem per ∂D_R

$$\partial D_R := \{p \in P \,:\, r_{o_N}(p) = R\}. \tag{2.6.3}$$

Enunciem ara un teorema que ens mostre les propietats d'aquesta distància extrínseca

Teorema 2.6.1. *Siga P^m una varietat riemanniana m-dimensional, siga $\varphi : P^m \to N^n$ una immersió isomètrica de P a la varietat n-dimensional N. Suposem que N poseeix un pol o_N i definim la funció distància extrínseca $r_{o_N} : P \to \mathbb{R}^+$ com hem vist anteriorment. Aleshores*

1. *r_{o_N} és C^∞ en $P - \varphi^{-1}(o_N)$.*
2. *$\varphi(D_R) \subset B^N_R(o)$ sent $B^N_R(o_N)$ la bola geodèsica en N de radi R centrada en o_N.*
3. *Les esferes extrínseques ∂D_R són subvarietats diferenciables de P, menys per a uns valors del radi de mesura de Lebesgue nul·la en $\mathbb{R}^+$.*
4. *$\|\nabla^P r_{o_N}\| \leq 1$*
5. *Si P és completa i existeix $o \in P$ tal que $\varphi(o) = o_N$, aleshores $B^P_R(o) \subseteq D_R$. On $B^P_R(o)$ representa la bola geodèsica en P centrada en o i de radi R.*
6. *Les boles extrínseques D_R no són, en general, connexes.*
7. *Si, a més a més, φ és una immersió isomètrica pròpia tenim que*
   (a) *D_R és un conjunt precompacte i ∂D_R és un compacte.*
   (b) *$\frac{d}{dt}\operatorname{Vol}(D_t) = \int_{\partial D_t} \frac{1}{\|\nabla^P r\|} d\sigma_t \geq \operatorname{Vol}(\partial D_t)$ per a (quasi) tot t.*

Demostració. Per al primer punt sols cal considerar que la funció distància $d^N_{o_N}$ és C^∞ en $N - \{o_N\}$. El segon punt és una conclusió de la definició de bola extrínseca (equació 2.6.2). El tercer punt és una conseqüència del teorema de la funció inversa per a valors regulars. Tot açò no pot ser aplicat en valors crítics, però pel teorema de Sard aquests formen un conjunt de mesura de Lebesgue nul·la en $\mathbb{R}^+$ (veure secció 2.4) .

Per al quart punt, cal tenir present que

$$\nabla^N r = \nabla^P r + (\nabla^P r)^\perp. \tag{2.6.4}$$

i que $\|\nabla^N r\| = \|\nabla^N d^N_{o_N}\| = 1$. Per tant

$$\begin{aligned} \|\nabla^P r\|^2 =& \langle \nabla^P r, \nabla^P r\rangle \\ =& \langle \nabla^N r - (\nabla^P r)^\perp, \nabla^N r - (\nabla^P r)^\perp\rangle \\ =& \|\nabla^N r\|^2 - 2\langle \nabla^N r, (\nabla^P r)^\perp\rangle + \|(\nabla^P r)^\perp\|^2 \\ =& \|\nabla^N r\|^2 - \|(\nabla^P r)^\perp\|^2 \\ \leq& \|\nabla^N r\|^2 = 1. \end{aligned} \tag{2.6.5}$$

Per al cinquè punt, tenint en compte que per ser P completa per a qualsevol $p \in P$ existeix una geodèsica $\gamma : [0, \rho_P(p)] \to P$ tal que la longitud de γ és igual a

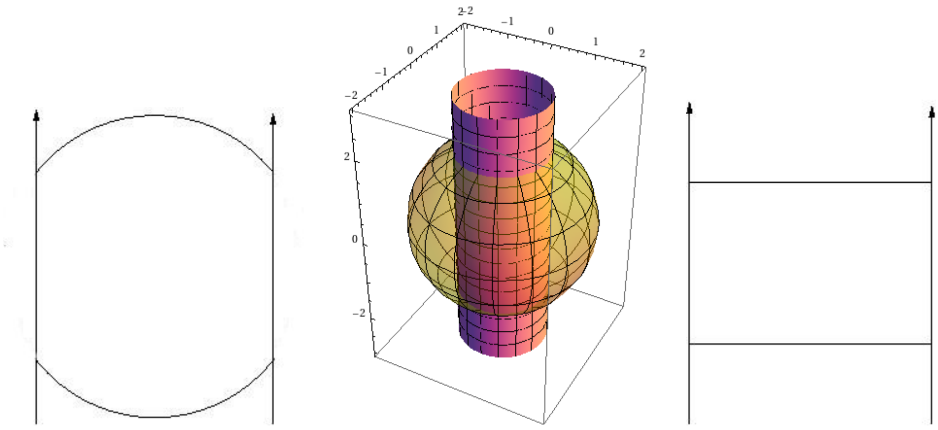


Figura 2.2: Continuant amb l'exemple de la figura 2.1 si superem el radi d'injectivitat les esferes geodèsiques són corbes amb angles per al cilindre, com veiem en la figura de l'esquerra. Però podem fer una immersió isomètrica del cilindre en $\mathbb{R}^3$ com veiem en la figura d'enmig. Utilitzant la distància des de l'origen en $\mathbb{R}^3$ obtenim esferes extrínseques completament suaus per al cilindre com veiem en la figura de la dreta.

la distància dist $^P(o,p)$, però aleshores $\varphi \circ \gamma$ és una corba unint o_N i $\varphi(p)$ aleshores per la definició de distància (veure definició 2.5.1)

$$r_{o_N}(p) = \text{dist}\,^N(o_N, \varphi(p)) \leq \text{dist}\,^P(o,p). \tag{2.6.6}$$

Finalment, per al setè punt sols cal aplicar la fórmula de la coàrea. Tenint en compte que les esferes extrínseques són els conjunts de nivell de la funció distància extrínseca i que les boles extrínseques per ser la immersió pròpia són conjunts precompactes. □

Nota 2.6.2. *En [Whi87] B. White demostra que tota superfície completa amb integral finita de la part negativa de la curvatura seccional és una superfície amb tipus topològic finit. En la demostració aplica el comportament del creixement de les esferes geodèsiques de la superfície. Com les esferes geodèsiques en aquests cas són corbes que en general no tenen perquè ser suaus en tots els punts necessita estimar el valor dels angles que es formen. Aquesta computació no és en general senzilla, de fet un any després B. White publica en [Whi88] una correcció al seu article basat en un comentari de Peter Li. Nosaltres en el capítol 3 apliquem la fórmula de Gauss-Bonnet però per a les boles extrínques suposant que existeix una immersió minimal completa des de la superfície a una forma espacial real de curvatura constant menor o igual a zero. Per la suavitat de la distància extrínseca , en el capítol 3 no necessitem preocupar-nos en absolut dels possibles angles de les corbes que formen les esferes extrínseques. Un exemple similar s'il·lustra en la figura 2.2 per al cas del cilindre.*

2.7 Varietats producte i espais models

Fins ara hem vist com tenint una immersió $\varphi : P \to N$ de la varietat P a la varietat N amb pol, podem construir la funció distància extrínseca i alguna de les seues propietats. En particular hem vist la construcció de les boles geodèsiques D_R i de les esferes geodèsiques ∂D_R. Ara allò que ens interessa és obtenir algunes propietats de D_R (o de ∂D_R) per poder extreure informació sobre P.

El nostre plantejament serà utilitzar teoremes de comparació entre D_R i la bola geodèsica de radi R en una varietat que ens servirà de model i de la qual devem conèixer les seues propietats.

Les varietats que utilitzarem com a model en els casos més senzills seran formes espacials reals de curvatura seccional constant, però en general necessitem una generalització de les superfícies de revolució que anomenarem espai model M_W^m i es construeix com un producte deformat (anomenat des d'ara endavant amb la paraula anglesa "warped") $M_W^m = I \times_W S_1^{m-1}$ d'un interval de la recta real I per l'esfera de radi unitat $m-1$ dimensional S_1^{m-1} amb funció "warped" W.

Veurem més endavant en els capítols destinats als resultats com escollir una funció W adient en relació a les restriccions de les curvatures seccionals radials de la varietat N i les restriccions a la curvatura mitja radial de la immersió $\varphi : P \to N$. En aquesta secció repassem les principals propietats de les varietats producte i la construcció d'un espai model.

Recordem que

Definició 2.7.1. *Siguen B, F dues varietat de dimensió n i m respectivament, anomenem $M = B \times F$ el producte directe de B i F com a espais topològics. L'espai M està format per parelles (x, y) on $x \in B$ i $y \in F$, i li podem associar de forma natural una estructura de varietat diferenciable. Així, si U i V són cartes sobre B i F respectivament, amb coordenades $x^1, \cdots, x^n$ i $y^1, \cdots, y^m$ aleshores $U \times V$ és una carta sobre M amb coordenades $x^1, \cdots, x^n, y^1, \cdots, y^m$. L'atles de totes aquestes cartes fan de M una varietat diferenciable.*

Per qualsevol punt $(x, y) \in M$ l'espai tangent $T_{(x,y)}M$ està identificat de forma natural am la suma directa $T_xB \oplus T_yF$. De fet, qualsevol $\mathbb{R}-$diferenciació $\xi \in T_xB$ pot ser considerada com una $\mathbb{R}-$diferenciació sobre funcions $f(x, y)$ de M mantenint fixa la variable y, o siga:

$$\xi(f) = \xi(f(\cdot, y)). \tag{2.7.1}$$

Açò identifica T_xB com un subespai de $T_{(x,y)}M$, i el mateix pot ser aplicat a T_yB. Seguint [Gri09] o [O'N83] es pot demostrar senzillament que la intersecció de T_xB i T_yF en $T_{(x,y)}M$ és $\{0\}$, que dim $T_{(x,y)}M = n + m$ i que

$$T_{(x,y)}M = T_xB \oplus T_yF. \tag{2.7.2}$$

Definim l'estructura mètrica de la varietat producte com

Definició 2.7.2. *Siguen (B, g_B) i (F, g_F) dues varietats riemannianes. Definim el tensor mètric g sobre $M = B \times F$ com la suma directa:*

$$g = \pi^*(g_B) + \sigma^*(g_F) \tag{2.7.3}$$

on π i σ són les projeccions de $B \times F$ sobre B i F respectivament.

És a dir, per a qualsevol punt $(x, y) \in M$, qualsevol dos vectors ξ i η en $T_{(x,y)}M$ es descomponen de forma única com:

$$\begin{aligned} \xi =& \xi_B + \xi_F \\ \eta =& \eta_B + \eta_F \end{aligned} \tag{2.7.4}$$

On ξ_B i η_B pertanyen a T_xB, i ξ_F i η_F pertanyen a T_yF, aleshores

$$\langle \xi, \eta \rangle_M = \langle \xi_B, \eta_B \rangle_B + \langle \xi_F, \eta_F \rangle_F \tag{2.7.5}$$

En les coordenades locals $x^1, \cdots, x, y^1, \cdots, y^m$ tenim:

$$g = \pi^*(g_B) + \sigma^*(g_F) = (g_B)_{ij} dx^i dx^j + (g_F)_{kl} dy^k dy^l \tag{2.7.6}$$

on s'aplica el criteri de sumació per a índex repetits.

Òbviament la varietat producte $M = B \times F$ amb la mètrica producte definida anteriorment esdevé una varietat riemanniana (M, g) que serà anomenada varietat riemanniana producte. Cal fer menció que la matriu del tensor g té forma de bloc

$$g = \begin{pmatrix} g_B & 0 \\ 0 & g_F \end{pmatrix} \tag{2.7.7}$$

que implica una forma similar per a la matriu inversa g^{-1} i que per al determinant

$$\det g = \det g_B \det g_F \tag{2.7.8}$$

Si ν_B i ν_F són les mesures riemannianes en B i F, respectivament, aleshores la mesura riemanniana ν de M ve donada per

$$\begin{aligned} d\nu &= \sqrt{\det g} dx^1 \cdots dx^n dy^1 \cdots dy^m \\ &= \left(\sqrt{\det g_B} dx^1 \cdots dx^n \right) \left(\sqrt{\det g_F} dy^1 \cdots dy^m \right) = d\nu_B \, d\nu_F. \end{aligned} \tag{2.7.9}$$

I, per tant, ν és el producte de les mesures ν_B i ν_F, que és,

$$\nu = \nu_B \times \nu_F \tag{2.7.10}$$

(veure [Gri09] per a la definició del producte de mesures).

Recordem, també les definicions dels *aixecaments horitzontals* (*verticals*) de funcions, vectors i camps vectorials

Definició 2.7.3. *(veure [O'N83] o [KN96] per exemple) Siga la varietat producte $M = B \times F$, siguen π i σ les projeccions $\pi : M \to B$ i $\sigma : M \to F$. Donada $f : B \to \mathbb{R}$ una funció definida sobre la varietat B anomenarem a $\overline{f} : B \times F \to \mathbb{R} = f \circ \pi$ el* aixecament horitzontal *de f a $B \times F$.*

Donat un vector $\eta \in T_pB$, anomenarem aixecament horitzontal *$\overline{\eta}$ de η en $(p, q) \in B \times F$ a l'únic vector en $T_{(p,q)}B \times F$ tal que $\pi_*(\overline{\eta}) = \eta$ i $\sigma_*(\overline{\eta}) = 0$.*

Si X és un camp vectorial sobre B l' aixecament horitzontal *de X a $B \times F$ serà el camp vectorial $\overline{X}$ tal que per a cada $(p, q) \in B \times F$ és l'* aixecament horitzontal *de X_p a $(p, q) \in B \times F$.*

Al conjunt de tots els aixecament horitzontals *de B el denotarem $\mathcal{L}(B)$ i de forma completament anàloga anomenarem $\mathcal{L}(F)$ al conjunt de tots els* aixecament verticals *de F.*

Un tema transversal en aquest treball és la comparació de les varietats amb pol i les seues subvarietats immerses amb espais models. Un espai model és una generalització de les superfícies de revolució fent ús del producte "warped".

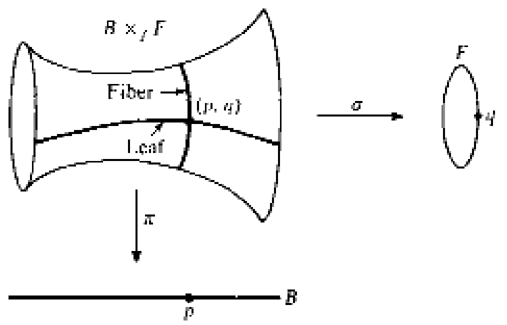
Com hem vist anteriorment en un producte $B \times F$ de dues varietats de Riemman B^n i F^m de dimensió m i n respectivament, el tensor mètric és $\pi^*(g_B) + \sigma^*(g_F)$ on π i σ són les projeccions de $B \times F$ sobre B i F respectivament. No obstant, podríem fer coses més interessants de la següent manera:

Definició 2.7.4. *(Veure [O'N83]) Siga (B, g_B) i (F, g_F) dues varietats de Riemann, i siga $f > 0$ una funció suau sobre B. El producte warped $M = B \times_f F$ serà la varietat producte $B \times F$ dotada de la mètrica:*

$$g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F) \tag{2.7.11}$$

En aquesta definició de producte "warped" anomenem a la varietat B la base de M i a F la fibra de M. El nostre objectiu serà expressar la geometria de M en termes de la funció "warped" i de les geometries de B i de F.

Les *fibres* $p \times F = \pi^{-1}(p)$ i les *fulles* ("leaves" en anglès) $B \times q = \sigma^{-1}(q)$ són subvarietats de M com s'observa en la següent figura (veure [O'N83]):



El producte "warped" té les següents propietats:

Proposició 2.7.5. *(veure [O'N83]) Siga $M = B \times_f F$ un producte "warped". Aleshores:*

1. *Per cada $q \in F$, l'aplicació $\pi|(B \times q)$ és una isometria sobre B.*

2. *Per cada $q \in B$, l'aplicació $\sigma|(p \times F)$ és una homotècia positiva sobre F amb factor d'escala $1/f(p)$ (una homotècia és un difeomorfisme $\psi : M \to N$ tal que $\psi^*(g_N) = c\, g_M$ per a certa constant c (veure [O'N83]))*

3. *Per cada $(p, q) \in M$, la fulla $B \times q$ i la fibra $p \times F$ són ortogonals en (p, q). És a dir els vectors tangents a la fulla (anomenats horitzontals) i els vectors tangents a la fibra (anomenats verticals) són ortogonals.*

4. *Siga la funció $h : B \to \mathbb{R}$, aleshores el gradient de l'aixecament horitzontal $h \circ \pi$ de h a $B \times_f F$ és l'aixecament horitzontal a $B \times_f F$ del gradient de h en B (és a dir $\pi_*(\nabla^M h \circ \pi) = \nabla^B h$ i $\sigma_*(\nabla^M h \circ \pi) = 0$).*

5. *Si* $X, Y \in \mathcal{L}(B)$ *i* $V, W \in \mathcal{L}(F)$ *aleshores:*

   (a) $\nabla^M_X Y \in \mathcal{L}(B)$ *és l'aixecament horitzontal de* $\nabla^B_X Y$.

   (b) $\nabla^M_X V = \nabla^M_V X = \frac{X(f)}{f} V$.

6. *Per la propietat 5(a) les fulles d'un producte "warped"* $B \times q$ *són subvarietats totalment geodèsiques. I per la propietat 5(b) les fulles d'un producte "warped" són subvarietats totalment umbilicals.*

7. *Siguen* U, V, $W \in \mathcal{L}(F)$ *aleshores*

$$R^M(V,W)U = R^F(V,W)U - \frac{\langle \nabla^M f, \nabla^M f \rangle}{f^2} \left(\langle V, U \rangle W - \langle W, U \rangle V \right).$$

   on R^M *i* R^F *denoten els tensors curvatura sobre les varietats* $M = B \times_f F$ *i* F *respectivament.*

En coordenades locals $x^1, \cdots, x^n, y^1, \cdots, y^m$ la mètrica g tindrà la següent expressió:

$$g = (g_B)_{ij} dx^i dx^j + f^2(x)(g_F)_{lk} dy^l dy^k \tag{2.7.12}$$

on s'aplica el criteri de sumació per a índex repetits.

La matriu de la mètrica g en el punt $(x, y) \in M = B \times_f F$ tindrà la forma de bloc

$$g = \begin{pmatrix} g_B & 0 \\ 0 & f^2(x) g_F \end{pmatrix} \tag{2.7.13}$$

i per tant el determinant serà

$$\det g = f^{2m}(x) \det g_B \det g_F \tag{2.7.14}$$

on recordem que m és la dimensió de F. I per tant la densitat de mesura riemanniana associada a la mètrica g ve donada per (veure [Gri09])

$$d\nu = f^m(x) d\nu_B d\nu_F \tag{2.7.15}$$

Un cas especialment interessant d'aquest tipus de producte és un espai model

Definició 2.7.6. *(Veure [Gri99],[GW79]) Anomenarem espai model* M^n_w *al producte "warped" d'un interval de la recta real* $I = (0, r_0)$ *amb l'esfera estàndard* $(n-1)$ *dimensional* S^{n-1}_1 *i funció "warped"* w*, és a dir:*

$$M^n_w = I \times_w S^{n-1}_1 \tag{2.7.16}$$

Definició 2.7.7. *Al valor* r_0 *donat en la definició anterior l'anomenarem* radi *de l'espai model* M^n_w.

Nota 2.7.8. *Podem estendre la base de l'espai model $B = (0, r_0)$ fins a $B = [0, r_0)$, però per a que la mètrica siga suau s'ha de garantir (veure [Pet98]) que:*

$$\begin{cases} w(0) = 0 \\ w'(0) = 1 \\ w^{(2n)}(0) = 0. \end{cases} \tag{2.7.17}$$

on $w^{(2n)}(0)$ són les derivades parelles de w avaluades en 0.

Proposició 2.7.9. *(Veure [Gri09]) Siga $M^n_w = [0, r_0) \times_w \mathbb{S}^{n-1}_1$ un espai model, aleshores:*

1. *Hi ha una carta sobre M^n_w que cobreix tot M^n_w, i la image d'aquesta carta en $\mathbb{R}^n$ és una bola*
$$B_{r_0} := \{x \in \mathbb{R}^n \,:\, |x| < r_0\} \tag{2.7.18}$$

2. *Si $r_0 = \infty$, l'espai model és una varietat amb pol, sent $\pi^{-1}(0)$ un pol de la varietat.*

3. *La mètrica g en coordenades polars (r, θ) en l'anterior carta té la forma*
$$g = dr^2 + w^2(r) g_{\mathbb{S}^{n-1}_1}, \tag{2.7.19}$$
on $w(r)$ és una funció positiva i suau definida en $(0, r_0)$ i $\mathbb{S}^{n-1}_1$ és l'esfera de radi 1 en $\mathbb{R}^n$.

Anomenem centre de l'espai model a

Definició 2.7.10. *Siga $M^n_w = [0, r_0) \times \mathbb{S}^{n-1}_1$ un espai model, el punt $o_w = \pi^{-1}(0)$ serà anomenat centre de l'espai model, sent π la projecció $\pi : [0, r_0) \times \mathbb{S}^{n-1}_1 \to [0, r_0)$.*

Com les fulles en un producte "warped" són subvarietats totalment geodèsiques (veure proposició 2.7.5) i un espai model és un tipus particular de producte "warped", la distància des de o_w a qualsevol punt $p \in M^n_w$ és justament $\pi(p)$ que anomenarem $r(p)$. L'esfera geodèsica $S^w_R = \partial B^w_R$ serà, per tant, la fibra $\pi^{-1}(R)$.

Aplicant la fórmula (2.7.15) que ens dóna la densitat de la mesura riemanniana d'un "warped" al cas particular d'un espai model obtenim:

$$d\nu = w^{n-1}(r) dr d\nu_{\mathbb{S}^{n-1}_1} \tag{2.7.20}$$

així el volum de la bola geodèsica de radi R és

$$\operatorname{Vol}(B^w_R) = \omega_n \int_0^R w^{n-1}(t) dt \tag{2.7.21}$$

on ω_n és el volum de $\mathbb{S}^{n-1}_1$. I el volum de l'esfera geodèsica S^w_R de radi R,

$$\operatorname{Vol}(S^w_R) = \omega_n w^{n-1}(R) \tag{2.7.22}$$

El vector unitari radial ortogonal a les esferes geodèsiques en l'espai model serà el gradient $\nabla^{M_w^n} r$, sent r la funció distància al centre del model o_w. Però com la funció distància al centre del model o_w és justament la projecció sobre la base $r(p) = \pi(p)$, per la propietat 4 de la proposició 2.7.5 tenim que el vector unitari normal a les esferes geodèsiques serà l'aixecament horitzontal de $\nabla^{[0,r_0)} r$ (i.e, $\frac{\partial}{\partial r}$, sent r la coordenada en $[0, r_0)$). Per tant, el vector ortogonal a les esferes geodèsiques (les fibres) serà el vector horitzontal donat per l'aixecament horitzontal anterior. I els plans radials de l'espai tangent (veure definició 2.5.9) seran aquells que el continguen.

En un espai model, la curvatura radial $K_w(\Pi)$ de M_w^n per a qualsevol $x \in M_w^n$ (i qualsevol pla radial Π, veure definició 2.5.10) és una funció que depèn única i exclusivament de $r(x)$, on r és la funció distància del model (M, o_w) relativa al seu centre o_w com es demostra en la següent proposició

Proposició 2.7.11. *Siga M_w^n un espai model $M_w^n = [0, r_0) \times_w S_1^{n-1}$ aleshores la curvatura seccional radial K_w de qualsevol punt $p \in M_w^n - \{o_w\}$ val*

$$K_w(p) = -\frac{w''(r(p))}{w(r(p))}. \tag{2.7.23}$$

Sent $r(p)$ la distància des de o_w fins a p, és a dir $\pi(p)$.

Demostració. Per demostrar la proposició ens cal calcular la curvatura seccional dels plans radials de l'espai tangent, és a dir la curvatura seccional dels subespais bidimensionals de l'espai tangent que continguen al vector radial. En l'explicació precedent a l'enunciat de la proposició hem vist que el vector radial és $\nabla^{M_w^n} r$, que és el vector horitzontal obtingut com l'aixecament l'horitzontal de $\frac{\partial}{\partial r}$.

Per a qualsevol punt $p \in M_w^n - \{o_w\}$ podem descompondre qualsevol vector $v \in T_pM$ de l'espai tangent T_pM de forma única com $v = v_H + v_V$ sent v_H tangent a la fulla (horitzontal, paral·lel al vector radial, ortogonal a l'esfera geodèsica $S_{r(p)}^w$) i v_H tangent a la fibra (vertical, ortogonal al vector radial, tangent a l'esfera geodèsica $S_{r(p)}^w$).

Com la base és de dimensió 1, els plans radials contenen al vector radial i a un altre vector V vertical (ortogonal al vector radial). Si apliquem ara, la definició de curvatura seccional al pla format per V i $\nabla^{M_w^n} r$ obtenim

$$\begin{aligned} K_w &= \frac{\langle R(V, \nabla^{M_w^n} r)\nabla^{M_w^n} r, V\rangle}{\|\nabla^{M_w^n} r\|^2\|V\|^2 - \langle \nabla^{M_w^n} r, V\rangle} \\ &= \frac{\langle \nabla_V \nabla_{\nabla^{M_w^n} r} \nabla^{M_w^n} r - \nabla_{\nabla^{M_w^n} r} \nabla_V \nabla^{M_w^n} r - \nabla_{[V, \nabla^{M_w^n} r]} \nabla^{M_w^n} r, V\rangle}{\|V\|^2} \end{aligned} \tag{2.7.24}$$

Però, pel fet de ser les fulles subvarietats totalment geodèsiques, $\nabla_{\nabla^{M_w^n} r} \nabla^{M_w^n} r = 0$ (ja que $\nabla^{M_w^n} r$ és l'aixecament del vector tangent a la geodèsica en $[0, r_0)$), i per ser V un vector vertical (veure [O'N83]) $[\nabla^{M_w^n} r, V] = 0$, per tant utilitzant la propietat

5(b) de 2.7.5

$$\begin{aligned}K_w &= -\frac{\langle \nabla_{\nabla^{M_w^n} r} \nabla_V \nabla^{M_w^n} r, V\rangle}{\|V\|^2} = -\frac{\langle \nabla_{\nabla^{M_w^n} r}(\frac{\nabla^{M_w^n} r(w)}{w} V), V\rangle}{\|V\|^2} \\ &= -\frac{\langle \nabla_{\nabla^{M_w^n} r}(\frac{w'}{w} V), V\rangle}{\|V\|^2} = -\frac{\frac{w''}{w}\langle V, V\rangle}{\|V\|^2} = -\frac{w''}{w}\end{aligned} \tag{2.7.25}$$

□

Si ens fixem en la curvatura mitja de les esferes geodèsiques podem obtenir la següent proposició

Proposició 2.7.12. *Siga $M_w^n : [0, r_0) \times_w \mathbb{S}_1^{n-1}$ un espai model. Aleshores la curvatura mitja η_w de les esferes geodèsiques S_R^w de radi R apuntant cap a dins val:*

$$\eta_w(R) = \frac{w'(R)}{w(R)}. \tag{2.7.26}$$

Demostració. Siga $p \in S_R^w$ i siguen X, Y dos vectors de $T_p S_R^w$. Tenint present que el vector unitari normal a les esferes geodèsiques apuntant cap a fora val $\nabla^{M_w^n} r$ i que l'esfera geodèsica és una hipersuperfície en M_w^n (codimensió 1), aleshores la segona forma fonamental B de la esfera S_R^w en el punt p per als vectors X, Y val

$$\begin{aligned}B(X, Y)_p =&\langle B(X, Y)_p, \nabla^{M_w^n} r\rangle \nabla^{M_w^n} r = \langle \nabla_X Y, \nabla^{M_w^n} r, \rangle_p \nabla^{M_w^n} r \\ =& -\langle Y, \nabla_X \nabla^{M_w^n} r, \rangle_p \nabla^{M_w^n} r.\end{aligned} \tag{2.7.27}$$

Aplicant la propietat 5(b) de la proposició 2.7.5 arribem a

$$\begin{aligned}B(X, Y)_p =& -\langle Y, \frac{\nabla^{M_w^n} r(w)}{w}(r)\rangle_p \nabla^{M_w^n} r \\ =& \left(-\eta_w(R) \nabla^{M_w^n} r\right) \langle X, Y\rangle.\end{aligned} \tag{2.7.28}$$

Recuperant així, l'afirmació 6 de la proposició 2.7.5 la qual ens deia que les fibres (en aquest cas les esferes geodèsiques) són subvarietats totalment umbilicals. Traçant l'expressió anterior arribem a que el vector curvatura mitja H_p per a qualsevol punt $p \in S_R^w$ val

$$H_p = -\eta_w(R) \nabla^{M_w^n} r. \tag{2.7.29}$$

Per tant el vector curvatura mitja apuntant cap a dins val

$$\langle -\eta_w(R) \nabla^{M_w^n} r, -\nabla^{M_w^n} r\rangle = \eta_w(R). \tag{2.7.30}$$

□

Finalment, en un espai model a part de la curvatura seccional radial podem estudiar la curvatura seccional dels plans continguts en el tangent a les fibres S_R^w

Proposició 2.7.13. *Siga $p \in M_w^n$ sent M_w^n un espai model amb $n > 2$. Siguen V, W linealment independents i tangents a l'esfera geodèsica S_R^w, aleshores*

$$K_p(V, W) = \frac{1}{w^2(R)} - \eta_w^2(R). \tag{2.7.31}$$

On $K_p(V, W)$ representa la curvatura seccional del $2-$pla format per V, W i $R = r(p)$

Demostració. Com M_w^m és un producte "warped" $M_w^n = [0, r_0) \times_w S_1^{n-1}$ on la direcció radial coincideix amb la direcció de les fulles i les direccions tangents a S_w^R són les direccions tangents a les fibres. Per la propietat 7 de la proposició 2.7.5 sabem que donats dos vectors V, W linealment independents i tangents a S_R^W, la curvatura seccional $K_p(V, W)$ del pla format per V, W en el punt $p \in M_w^m$ val

$$\begin{aligned} K_p(V, W) =& \frac{\langle R_{VW} V, W\rangle}{\|V\|^2\|W\|^2 - \langle V, W\rangle^2} \\ =& \frac{\langle R_{VW}^{S_1^{m-1}} V - \frac{\langle \nabla w, \nabla w\rangle}{w^2} \left(\langle V, V\rangle W - \langle W, V\rangle V\right), W\rangle}{\|V\|^2\|W\|^2 - \langle V, W\rangle^2} \\ =& \frac{\langle R_{VW}^{S_1^{m-1}} V, W\rangle}{\|V\|^2\|W\|^2 - \langle V, W\rangle^2} - \frac{\langle \nabla^{M_w^n} w, \nabla^{M_w^n} w\rangle}{w^2}. \end{aligned} \tag{2.7.32}$$

On $R^{S_1^{m-1}}$ denota el tensor curvatura sobre l'esfera unitat i hem fet la identificació $V = \sigma_*(V)$ (i $W = \sigma_*(W)$) sent σ la projecció $\sigma : M_w^n \to S_1^{m-1}$. Tenint en compte la definició de la mètrica en un producte "warped", sabent que la curvatura seccional sobre l'esfera unitat és 1 i que $\|\nabla^{M_w^n} r\| = 1$, tenim que

$$\begin{aligned} K_p(V, W) =& \frac{\langle R_{VW}^{S_1^{n-1}} V, W\rangle_{S_1^{n-1}}}{w^2 \left(\|V\|^2_{S_1^{n-1}} \|W\|^2_{S_1^{n-1}} - \langle V, W\rangle^2_{S_1^{n-1}}\right)} - \frac{\langle \nabla^{M_w^n} w, \nabla^{M_w^n} w\rangle}{w^2} \\ =& \frac{1}{w^2} - \frac{(w')^2 \langle \nabla^{M_w^n} r, \nabla^{M_w^n} r\rangle}{w^2} \\ =& \frac{1}{w^2} - \eta_w^2. \end{aligned} \tag{2.7.33}$$

□

Si elegim amb molta cura la funció w del nostre espai model obtenim una forma espacial real

Proposició 2.7.14. *Siga $M_{w_b}^n = [0, r_0(b)) \times_{w_b} \mathbb{S}_1^{n-1}$ un espai model on*

$$w_b(r) = \begin{cases} \frac{1}{\sqrt{-b}} sinh(\sqrt{-b}\, r) \;\; si\ b < 0 \\ r \;\; si\ b = 0 \\ \frac{1}{\sqrt{b}} sin(\sqrt{b}\, r) \;\; si\ b > 0 \end{cases} \tag{2.7.34}$$

i

$$r_0(b) = \begin{cases} \infty \ si\ b \leq 0 \\ \frac{\pi}{\sqrt{b}} \ si\ b > 0 \end{cases} . \tag{2.7.35}$$

Aleshores $M^n_{w_b}$ és una varietat simplement connexa amb curvatura seccional constant igual b. És a dir, $M^n_{w_b}$ és la forma espacial real $\mathbb{K}^n(b)$.

Demostració. Per a la demostració d'aquest teorema sols cal tenir present les propietats dels espais models (proposició 2.7.9), computar les curvatures seccionals dels plans radials per la proposició 2.7.11, les curvatures seccionals dels plans tangents a les fibres per la proposició 2.7.13, i adonar-se de que són constants i iguals a b per a tot valor de R. □

El volum de les esferes i de les boles geodèsiques de radi R en la forma espacial real $\mathbb{K}^n(b)$, que denotarem com $\mathrm{Vol}(S^{b,n-1}_R)$ i $\mathrm{Vol}(B^{b,n}_R)$ respectivament, valdrà:

$$\begin{aligned} \mathrm{Vol}(S^{b,n-1}_R) &= \omega_n w^{n-1}_b(R) \\ \mathrm{Vol}(B^{b,n}_R) &= \omega_n \int_0^R w^{n-1}_b(t)dt. \end{aligned} \tag{2.7.36}$$

Nota 2.7.15. *Aplicant l'expressió del volum de les boles geodèsiques a la forma espacial $\mathbb{K}^n(1)$ de curvatura constant 1 ($w(r) = sin\ (r)$) obtenim*

$$\omega_{n+1} = \mathrm{Vol}(B^{1,n}_\pi) = \omega_n \int_0^\pi sin^{n-1}(t)dt \tag{2.7.37}$$

sabent que $w_2 = 2\pi$, iterant aquesta fórmula podem obtenir l'expressió de w_n que no és altra que

$$w_n = 2\frac{\pi^{n/2}}{\Gamma(n/2)} \tag{2.7.38}$$

veure [Gri09], o [Gra04] per a més detalls.

Finalment, la curvatura mitja $h_b(R)$ de les esferes geodèsiques $S^{b,n-1}_R$ de radi R apuntant cap a dins en la forma espacial real $\mathbb{K}^n(b)$ de curvatura seccional constant b, serà

$$h_b(R) := \eta_{w_b}(R) = \frac{w'_b(R)}{w_b(R)} = \begin{cases} \sqrt{-b}\,\mathrm{cotanh}(\sqrt{-b}R) \text{ si } b < 0 \\ 1/R \text{ si } b < 0 \\ \sqrt{b}\,\mathrm{cotan}(\sqrt{b}R) \text{ si } b > 0 \end{cases} \tag{2.7.39}$$

Nota 2.7.16. *Per acabar aquesta secció recordem breument algunes propietats de les formes espacials reals:*

1. *Axioma de lliure mobilitat (veure [Sak96]): Per qualsevol parella de punts $p, p' \in \mathbb{K}^n(b)$ i qualsevol parella de bases ortonormals $\{\xi_i\}^n_{i=1}$ i $\{\xi'_i\}^n_{i=1}$ en p, p', respectivament, existeix una isometria des d'un entorn de p sobre un entorn de p', que transforma p en p' i $\{\xi_i\}^n_{i=1}$ en $\{\xi'_i\}^n_{i=1}$.*

2. *$\mathbb{K}^n(b)$ és un espai homogeni (veure [Sak96]).*

3. *El radi d'injectivitat $i(\mathbb{K}^n(b)) = \frac{\pi}{\sqrt{b}}$ si $b > 0$ i és infinit $i(\mathbb{K}^n(b)) = \infty$ en qualsevol altre cas ($b \leq 0$).*

4. *Axioma del pla (veure [Sak96]): Siga W un subespai $k-$dimensional de $T_p\mathbb{K}^n(b)$, on $p \in \mathbb{K}^n(b)$, aleshores $S := exp_p W \cap B_\epsilon(0_p)$ és una subvarietat totalment geodèsica de $\mathbb{K}^n(b)$ per a $0 < \epsilon < i_p(\mathbb{K}^n(b))$.*

El laplacià i el hessià sobre varietats producte i espais models

Si estudiem el hessià de funcions sobre productes "warped" podem obtenir que

Proposició 2.7.17. *Siga $M = B \times_w F$ un producte "warped", siga $\overline{f} : M \to \mathbb{R}$ l'aixecament de la funció $f : B \to \mathbb{R}$ ($\overline{f} = f \circ \pi$) aleshores per al hessià $Hess^M \overline{f}$ de $\overline{f}$ tenim*

$$Hess^M \overline{f}(X,Y) = Hess^B f(X_H, Y_H) + \frac{\nabla^B f(w)}{w} \langle X_V, Y_V \rangle. \tag{2.7.40}$$

On hem fet ús de la propietat $T_pM = T_{\pi(p)}B \oplus T_{\sigma(p)}F$ sent X_H (resp Y_V) la component de X (resp Y) en $T_{\pi(p)}B$ i X_V (resp Y_V) la component de X (resp Y) en $T_{\sigma(p)}F$

Demostració. Com, $X = X_H + X_V$,

$$\text{Hess}^M \overline{f}(X,Y) = \langle \nabla^M_X \nabla^M \overline{f}, Y \rangle = \langle \nabla^M_{X_H} \nabla^M \overline{f}, Y \rangle + \langle \nabla^M_{X_V} \nabla^M \overline{f}, Y \rangle \tag{2.7.41}$$

Per ser $\overline{f} = f \circ \pi$ aleshores $\nabla^M \overline{f} \in \mathcal{L}(B)$ (propietat 4 de la proposició 2.7.5), per tant aplicant les propietats de la connexió en el producte "warped" (propietat 5 de la proposició 2.7.5)

$$\begin{aligned} \text{Hess}^M \overline{f}(X,Y) &= \langle \nabla^M_{X_H} \nabla^M \overline{f}, Y_H \rangle + \frac{\nabla^M \overline{f}(w)}{w} \langle X_V, Y_V \rangle \\ &= \langle \nabla^B_{X_H} \nabla^B f, Y_H \rangle_B + \frac{\nabla^B f(w)}{w} \langle X_V, Y_V \rangle \\ &= \text{Hess}^B (X_H, Y_H) + \frac{\nabla^B f(w)}{w} \langle X_V, Y_V \rangle. \end{aligned} \tag{2.7.42}$$

□

Si apliquem aquest resultat al cas concret d'un espai model obtenim

Corol·lari 2.7.18. *Siga $M^n_w = [0, r_0) \times_w S^{n-1}_1$ un espai model, siga $r(p) =$ dist (o_w, p) la funció distància al centre del model o_w. Aleshores per al hessià de la funció distància tenim*

$$\text{Hess}^{M^n_w} r(X,Y) = (\eta_w \circ r)(p) \left(\langle X, Y \rangle - \langle X, \nabla^{M^n_w} r \rangle \langle Y, \nabla^{M^n_w} r \rangle \right). \tag{2.7.43}$$

Demostració. Com la funció distància és $r(p) = \pi(p)$, la base és de dimensió 1 i $\nabla^{M^n_w} r$ és horitzontal, tenim que per qualsevol camp vectorial X de M^n_w:

$$\begin{aligned} X_H =&\langle X, \nabla^{M^n_w} r\rangle \nabla^{M^n_w} r \\ X_V =&X - \langle X, \nabla^{M^n_w} r\rangle \nabla^{M^n_w} r \end{aligned} \tag{2.7.44}$$

Per tant aplicant la proposició anterior

$$\begin{aligned} \mathrm{Hess}^{M^n_w} r(X,Y) &= \frac{w'}{w}\langle X_V, Y_V\rangle \\ &= (\eta_w \circ r)\left(\langle X,Y\rangle - \langle X, \nabla^{M^n_w} r\rangle\langle Y, \nabla^{M^n_w} r\rangle\right). \end{aligned} \tag{2.7.45}$$

□

Aplicant la proposició 2.2.8 al resultat anterior obtenim

Corol·lari 2.7.19. *Siga* $M^n_w = [0, r_0) \times_w S^{n-1}_1$ *un espai model, siga* $r(p) = dist\,(o_w, p)$ *la funció distància al centre del model* o_w*, donada qualsevol funció suau* $F : \mathbb{R} \to \mathbb{R}$*, obtenim*

$$\begin{aligned} &\mathrm{Hess}^{M^n_w} F \circ r(X,Y) \\ &= (F'' \circ r)\langle X, \nabla^{M^n_w} r\rangle\langle Y, \nabla^{M^n_w} r\rangle \\ &+ (F' \circ r)(\eta_w \circ r)(p)\left(\langle X,Y\rangle - \langle X, \nabla^{M^n_w} r\rangle\langle Y, \nabla^{M^n_w} r\rangle\right) \end{aligned} \tag{2.7.46}$$

Si prenem la traça en aquesta darrera equació arribem a la següent expressió per al laplacià de funcions radials en un espai model

Corol·lari 2.7.20. *Siga* $M^n_w = [0, r_0) \times_w S^{n-1}_1$ *un espai model, siga* $r(p) = dist\,(o_w, p)$ *la funció distància al centre del model* o_w*, donada qualsevol funció suau* $F : \mathbb{R} \to \mathbb{R}$*, obtenim*

$$\Delta^{M^n_w} F \circ r(X,Y) = (F'' \circ r)(p) + (F' \circ r)(\eta_w \circ r)(p). \tag{2.7.47}$$

Nota 2.7.21. *El nostre camí per arribar a l'equació (2.7.47) en un espai model parteix de l'expressió del hessià en l'espai model per a després traçar el hessià i obtenir així el laplacià. No obstant, en [Gri09] s'arriba a l'equació (2.7.47) partint de la forma més general que adopta el laplacià d'una funció* $h : M \to \mathbb{R}$ *en un producte "warped"* $M = B \times_w F$

$$\Delta^M h = \Delta^B h + m\langle \nabla^B ln\ w, \nabla^B h\rangle_B + \frac{1}{w^2}\Delta^F h. \tag{2.7.48}$$

Que en el cas particular d'espais models i funcions radials recuperem l'equació (2.7.47).

Nota 2.7.22. *L'equació (2.7.47) implica en particular que*

$$\Delta^{M^n_w} r = (n-1)(\eta_w \circ r). \tag{2.7.49}$$

Com les esferes geodèsiques són hipersuperfícies de M_w^n amb vector unitari normal $\nabla^{M_w^n} r$, l'equació anterior ens aporta una altra forma d'obtenir el vector curvatura mitja de l'esfera S_R^w en el punt $p \in S_R^w$ (veure [dC92] per a relació entre la curvatura mitja d'una hipersuperfície i la divergència del vector normal)

$$\begin{aligned} H_p =& \frac{-1}{n-1}\left(div\ \nabla^{M_w^n} r\right)_p \nabla^{M_w^n} r \\ =& \frac{-\Delta^{M_w^n} r(p)}{n-1} \nabla^{M_w^n} r = -\eta_w(R) \nabla^{M_w^n} r. \end{aligned} \tag{2.7.50}$$

recuperant així l'equació 2.7.29.

2.8 Comparacions per al hessià: el teorema de Greene-Wu

Ara ja tenim totes les tècniques presentades per poder relacionar el hessià de la funció distància amb la curvatura seccional dels plans radials. Per fer açò utilitzarem el següent teorema de comparació donat per Greene-Wu (veure també [JK81])

Teorema 2.8.1. *(Veure [GW79], teorema A) Siguen (M, o) i (N, p) dues varietats amb pol tals que $\dim M \leq \dim N$. Siguen $\gamma_1 : [0, b] \to M$ i $\gamma_2 : [0, b] \to N$ dues geodèsiques normals amb $\gamma_1(0) = o$ i $\gamma_2(0) = p$. Suposem que per a tot $t \in [0, b]$,*

$$\text{curvatura seccional radial de } N \text{ a } \gamma_2(t) \geq \text{curvatura seccional radial de } M \text{ a } \gamma_1(t) \tag{2.8.1}$$

aleshores per a qualsevol funció f no decreixent en $[0, \infty)$

$$\mathrm{Hess}^N_{\gamma_2(t)}(f \circ \rho_N)(X, X) \leq \mathrm{Hess}^M_{\gamma_1(t)}(f \circ \rho_M)(Y, Y) \tag{2.8.2}$$

on ρ_N i ρ_M és la funció distància a o i p en N i M respectivament, X i Y són vectors de $T_{\gamma_2(t)}N$ i $T_{\gamma_1(t)}M$ respectivament tals que $|X| = |Y|$ i $\langle X, \nabla^N r\rangle = \langle Y, \nabla^M r\rangle$.

Aquest teorema el podem concretar fent comparacions amb espais models

Corol·lari 2.8.2. *Siga $N = N^n$ una varietat amb pol o. Suposem que cada curvatura seccional o-radial en $x \in N \setminus \{o\}$ està fitada per baix (respectivament, per dalt) per:*

$$K_{o,N}(\sigma_x) \geq\ (\leq)\ -\frac{w''(r(x))}{w(r(x))}$$

per a cada pla radial $\sigma_x \in T_xN$ a distància $r(x) = \mathrm{dist}_N(o, x)$ des d' o en N, sent $w : \mathbb{R} \to \mathbb{R}$ una funció suau. Aleshores el hessià de la funció distància en N satisfà

$$\mathrm{Hess}^N(r(x))(X, X) \leq\ (\geq)\ \eta_w(r(x))\left(|X|^2 - \langle \nabla^N r(x), X\rangle_N^2\right) \tag{2.8.3}$$

Per a tot vector X en T_xN, sent $\eta_w(r) = \frac{w'(r)}{w(r)}$.

Demostració. Escollim la dimensió n d'un espai model M^n_w i un camp vectorial Y en M^n_w de forma adient per poder aplicar-hi el teorema de Greene-Wu. Per al camp vectorial açò implica que, $\langle \nabla^{M^n_w} r(y), Y\rangle_{M^n_w} = \langle \nabla^N r(x), Y\rangle_N$ i $|Y| = |X|$ per a $r(y) = r(x)$. Aplicant ara el teorema de Greene-Wu, i el corol·lari 2.7.18 obtenim

$$\begin{aligned} \operatorname{Hess}^N(r(x))(X,X) \leq\ (\geq)\ & \operatorname{Hess}^{M^n_w}(r(y))(Y,Y) \\ &= \eta_w(r(x))\left(|Y|^2 - \langle \nabla^M r(y), Y\rangle^2_M\right) \\ &= \eta_w(r(x))\left(|X|^2 - \langle \nabla^N r(x), X\rangle^2_N\right) \end{aligned} \tag{2.8.4}$$

□

En el context que estàvem estudiant, suposem una immersió isomètrica $\varphi : P \to N$ entre la varietat P^m i la varietat N^n i suposem que la varietat N^n posseeix un pol $o_N \in N$. Si apliquem el corol·lari 2.3.10 a $F \circ r$ on $F : \mathbb{R} \to \mathbb{R}$ i r és la distància definida en N, $r(x) = \operatorname{dist}_N(o_N, x)$ obtenim

$$\begin{aligned} \operatorname{Hess}^P F \circ r|_P(X,Y)|_p = & F''(r(p))\langle \nabla^P r, X\rangle\langle \nabla^P r, Y\rangle \\ & + F'(r(p))\left(\operatorname{Hess}^N r(X,Y) + \langle \nabla^N r, B^P(X,Y)\rangle\right) \end{aligned} \tag{2.8.5}$$

així tenint en compte el corol·lari 2.8.2 arribem al següent teorema

Teorema 2.8.3. *Siga $\varphi : P^m \to N^n$ una immersió isomètrica entre la varietat P^m i la varietat N^n i suposem que la varietat N^n posseeix un pol $o_N \in N$. Denotarem amb $M = M^m_w$ un $w-$espai model amb centre o_w.*

1. *Si suposem que cada curvatura seccional o_N-radial en $x \in N \setminus \{o_N\}$ està fitada per per dalt per les curvatures seccional o_w-radials de M^m_w com segueix:*

$$K_{o,N}(\sigma_x) \leq -\frac{w''(t)}{w(t)}$$

per a cada pla radial $\sigma_x \in T_xN$ a distància $t = r(x) = \operatorname{dist}_N(o_N, x)$ des de o_N en N. Siga $F : \mathbb{R} \to \mathbb{R}$ una funció suau tal que $F' \geq 0$ (respectivament, $F' \leq 0$), aleshores per $F \circ r : P \to \mathbb{R}$, sent r la funció distància extrínseca, tenim que

$$\begin{aligned} \operatorname{Hess}^P F \circ r(X,X) \geq (\leq) & F''(t)\langle \nabla^N r, X\rangle^2 + \\ & F'(t)\left\{\eta_w(t)\left(|X|^2 - \langle \nabla^N r, X\rangle^2_N\right)\right. \\ & \left. + \langle \nabla^N r, B^P(X,X)\rangle\right\}. \end{aligned} \tag{2.8.6}$$

2. *Suposem que cada curvatura seccional o_N-radial en $x \in N \setminus \{o\}$ està fitada per per baix per les curvatures seccional o_w-radials de M^m_w com segueix:*

$$K_{o,N}(\sigma_x) \geq -\frac{w''(t)}{w(t)}$$

per a cada pla radial $\sigma_x \in T_xN$ a distància $t = r(x) = \mathrm{dist}_N(o,x)$ des de o_N en N. Siga $F : \mathbb{R} \to \mathbb{R}$ una funció suau tal que $F' \geq 0$ ($F' \leq 0$), aleshores per a la restricció en P de $F \circ r : P \to \mathbb{R}$ tenim

$$\begin{aligned}\mathrm{Hess}^P F \circ r(X,X) \leq (\geq) &F''(t)\langle \nabla^P r, X\rangle^2 \\ &+ F'(t)\left\{\eta_w(t)\left(|X|^2 - \langle \nabla^N r, X\rangle_N^2\right)\right. \\ &\left.+\langle \nabla^N r, B^P(X,X)\rangle\right\}.\end{aligned} \tag{2.8.7}$$

Per a tot vector X en T_xN

Prenent la traça de les expressions anteriors en una base ortonormal de T_pP obtenim el següent corol·lari

Corol·lari 2.8.4. *Siga N^n una varietat amb pol o, siga $\varphi : P^m \to N^n$ una immersió isomètrica. Siga M^m_w un $w-$model amb centre o_w. Aleshores tenim les següents desigualtats per al laplacià de la distància modificada $F \circ r : P \longrightarrow \mathbb{R}$:*

(i) Suposem que la curvatura seccional o-radial en $x \in N - \{o\}$ està fitada per la curvatura seccional o_w-radial en M^m_w com segueix:

$$\mathcal{K}(\sigma(x)) \,=\, K_{o,N}(\sigma_x) \geq -\frac{w''(t)}{w(t)} \quad . \tag{2.8.8}$$

On $t = r(x)$. Aleshores si $F' \leq 0$, (respectivament, $F' \geq 0$):

$$\begin{aligned}\Delta^P(F\circ r) \,\geq (\leq)\, &\left(F''(t) - F'(t)\eta_w(t)\right)\|\nabla^P r\|^2 \\ &+ mF'(t)\left(\eta_w(t) + \langle\, \nabla^N r,\, H_P\,\rangle\right) \quad ,\end{aligned} \tag{2.8.9}$$

on H_P representa el vector curvatura mitja de la immersió de P en N.

(ii) Suposem que la curvatura seccional o-radial en $x \in N - \{o\}$ està fitada per la curvatura seccional o_w-radial en M^m_w com segueix:

$$\mathcal{K}(\sigma(x)) \,=\, K_{o,N}(\sigma_x) \leq -\frac{w''(t)}{w(t)} \quad . \tag{2.8.10}$$

On $t = r(x)$. Aleshores si $F' \leq 0$, (respectivament, $F' \geq 0$):

$$\begin{aligned}\Delta^P(F\circ r) \,\leq (\geq)\, &\left(F''(t) - F'(t)\eta_w(t)\right)\|\nabla^P r\|^2 \\ &+ mF'(t)\left(\eta_w(t) + \langle\, \nabla^N r,\, H_P\,\rangle\right) \quad ,\end{aligned} \tag{2.8.11}$$

on H_P representa el vector curvatura mitja de la immersió de P en N.

2.9 Comparacions per al volum i el diametre

En aquesta secció repassarem breument les relacions que existeixen entre la curvatura, el volum i el diàmetre i que seran utilitzades en el capítol 5. Per fixar notació recordem que

Definició 2.9.1. *[Cha93] Siga M una varietat riemaniana, per a $p \in M$ definim el* tensol de Ricci *$Ric\ : T_pM \times T_pM \to \mathbb{R}$ per*

$$Ric\ (u, v) = traça\ (w \to R(u, w)v), \tag{2.9.1}$$

sent R el tensor curvatura i u, v, w vectors en T_pM.

En particular, tenim per a qualsevol base ortonormal de T_pM, $\{e_1, \cdots, e_n\}$,

$$\text{Ric}\ (u, v) = \sum_{i=1}^{n} \langle R(u, e_i)v, e_i \rangle \tag{2.9.2}$$

Amb açò, donat qualsevol vector $v \in T_pM$, si prenem una base ortonormal $\{e_1, \cdots, e_n\}$ de T_pM amb $e_n = v/|v|$; aleshores

$$\text{Ric}\ (v, v) = \left(\sum_{i=1}^{n-1} K(e_i, v) \right) |v|^2. \tag{2.9.3}$$

Sent $K(e_i, v)$ la curvatura seccional del subespai bidimensional de T_pM generat pels vectors v i e_i.

Recordem també que definim el diàmetre d'una varietat de Riemann com

Definició 2.9.2. *(Veure [Jos02] per exemple) Definim el diàmetre d'una varietat riemanniana M com:*

$$diam(M) = \sup_{p,q \in M} d(p, q), \tag{2.9.4}$$

on $d(\cdot, \cdot)$ denota la funció distància en M.

Amb aquestes definicions enunciem el següent teorema

Teorema 2.9.3. *(Teorema de Bonnet-Myers) (veure [Cha84],[Cha93],[Jos02]) Siga M una varietat riemanniana amb curvatura de Ricci tal que per a qualsevol $X \in TM$ existeix $b > 0$ complint que*

$$Ric(X, X) \geq b(n-1)|X|^2. \tag{2.9.5}$$

Aleshores:

1. *M és compacta.*
2. *M té grup fonamental finit.*
3. *I el diàmetre de M satisfà*

$$diam(M) \leq \pi/\sqrt{b}. \tag{2.9.6}$$

Nota 2.9.4. *Com l'esfera $\mathbb{S}^n(b)$ $n-$dimensional de curvatura seccional constant b, té curvatura de Ricci $Ric(v, v) = (n-1)b|v|^2$, i diàmetre $diam(\mathbb{S}^n(b)) = \pi/\sqrt{b}$, la desigualtat 2.9.6 del teorema anterior el que ens està dient és que si una varietat riemanniana M té curvatura de Ricci no menor que la curvatura de Ricci d'una esfera $\mathbb{S}^n(b)$), aleshores el diàmetre de M és com a màxim igual al diàmetre de $\mathbb{S}^n(b)$).*

Ara relacionarem la curvatura de Ricci amb el volum pel següent teorema

Teorema 2.9.5. *(Teorema de Bishop-Gromov)(veure [Cha84],[Cha93],[Sak96]) Siga M una varietat riemanniana amb curvatura de Ricci tal que per a qualsevol $X \in TM$ existeix $b \in \mathbb{R}$ complint que*

$$Ric(X,X) \geq b(n-1)|X|^2. \tag{2.9.7}$$

Aleshores, el volum de les boles geodèsiques B_R^M de radi R és menor o igual que el volum de les boles geodèsiques $B_R^{b,n}$ del mateix radi R en la forma espacial real $\mathbb{K}^n(b)$ de curvatura seccional constant b, és a dir:

$$\mathrm{Vol}(B_R^M) \leq \mathrm{Vol}(B_R^{b,n}) \tag{2.9.8}$$

Compint-se la igualtat si, i sols si, les dues boles geodèsiques són isomètriques. El quocient del volum de les boles geodèsiques

$$\frac{\mathrm{Vol}(B_R^M)}{\mathrm{Vol}(B_R^{b,n})} \tag{2.9.9}$$

és una funció decreixent respecte a R.

Si a més a més $b > 0$, aleshores

$$\mathrm{Vol}(M) \leq \frac{\mathrm{Vol}(S_1^{0,n})}{b^{n/2}}. \tag{2.9.10}$$

Amb igualtat si i sols si M és isomètrica a l'esfera $n-$dimensional $\mathbb{S}^n(b)$ de curvatura seccional constant b.

Nota 2.9.6. *Com l'esfera $\mathbb{S}^n(b)$ $n-$dimensional de curvatura seccional constant b, té volum $\mathrm{Vol}(\mathbb{S}^n(b)) = \frac{\mathrm{Vol}(S_1^{0,n})}{b^{n/2}}$, la desigualtat 2.9.10 del teorema anterior el que ens està dient és que si una varietat riemanniana M té curvatura de Ricci no menor que la curvatura de Ricci d'una esfera $\mathbb{S}^n(b)$), aleshores el volum de M no pot ser major que el volum de l'esfera $\mathbb{S}^n(b)$). Sent tots dos volum iguals tan sols en cas que M siga isomètrica a $\mathbb{S}^n(b)$).*

3

Isoperimetria extrínseca i compactificació de superfícies minimals de l'espai euclidià i de l'espai hiperbòlic

- Article publicat en Arxiv.org com *arXiv:1011.5380v2 [math.DG]*.
- Article acceptat per a la seua publicació en *Israel Journal of Mathematics*.

EXTRINSIC ISOPERIMETRY AND COMPACTIFICATION OF MINIMAL SURFACES IN EUCLIDEAN AND HYPERBOLIC SPACES

VICENT GIMENO# AND VICENTE PALMER*

ABSTRACT. We study the topology of (properly) immersed complete minimal surfaces P^2 in Hyperbolic and Euclidean spaces which have finite total extrinsic curvature, using some isoperimetric inequalities satisfied by the extrinsic balls in these surfaces, (see [12]). We present an alternative and partially unified proof of the Chern-Osserman inequality satisfied by these minimal surfaces, (in $\mathbb{R}^n$ and in $\mathbb{H}^n(b)$), based in the isoperimetric analysis above alluded. Finally, we show a Chern-Osserman type equality attained by complete minimal surfaces in the Hyperbolic space with finite total extrinsic curvature.

1. INTRODUCTION

Let us consider P^2 be a complete and minimal surface immersed in $\mathbb{R}^n$ and with finite total curvature $\int_P K^P d\sigma < \infty$, being K^P the Gauss curvature of the surface. Then we have the following equality (resp. inequality), known as the *Chern-Osserman formula*, (see [1], [3] and [8]):

$$(1.1)\quad -\chi(P) = \frac{1}{4\pi}\int_P \|B^P\|^2 d\sigma - \mathrm{Sup}_r \frac{\mathrm{Vol}(P^2\cap B_r^{0,n})}{\mathrm{Vol}(B_r^{0,2})} \leq \frac{1}{4\pi}\int_P \|B^P\|^2 d\sigma - k(P)$$

where $\chi(P)$ is the Euler characterisitic of P, k is its number of ends, B^P is the second fundamental foorm of P in $\mathbb{R}^n$ and $B_r^{b,n}$ denotes the geodesic r-ball in the simply connected real space form $\mathbb{K}^n(b)$.

To have finite total scalar (extrinsic) curvature $\int_P \|B^P\|^2 d\sigma < \infty$ is equivalent to the finiteness of the total Gaussian curvature (the original assumption in [3]) when the surface is minimal and immersed in $\mathbb{R}^n$. From this point of view, it is natural to wonder if it is possible to stablish a Chern-Osserman inequality (or equality) for complete minimal surfaces with finite total extrinsic curvature (properly) immersed in the hyperbolic space. This question has been addressed by Q. Chen and Y. Cheng in the papers [4] and [5]. They proved, for a complete minimal surface P^2 (properly) immersed in $\mathbb{H}^n(b)$ and such that $\int_P \|B^P\| d\sigma < \infty$, that $\mathrm{Sup}_r \frac{\mathrm{Vol}(P^2\cap B_r^{-1,n})}{\mathrm{Vol}(B_r^{-1,2})} < \infty$ and the following version of the Chern-Osserman Inequality, in terms of the volume growth of the extrinsic balls:

$$(1.2)\qquad -\chi(P) \leq \frac{1}{4\pi}\int_P \|B^P\|^2 d\sigma - \mathrm{Sup}_r \frac{\mathrm{Vol}(P^2\cap B_r^{-1,n})}{\mathrm{Vol}(B_r^{-1,2})}$$

The proofs given by these authors are different for those for the Euclidean case, and rely heavily on the properties of the hyperbolic functions.

We present in this paper a partial unification of the proof of the Chern-Osserman inequality (in terms of the volume growth) for complete minimal surfaces with finite total extrinsic curvature immersed in Euclidean or Hyperbolic spaces. This partial unification is based in obtaining estimates for the Euler characteristic of the extrinsic balls (given in

2000 *Mathematics Subject Classification.* Primary 53C20 ; Secondary 53C42, 49Q05.
Key words and phrases. Area growth, minimal surfaces, Chern-Osserman inequality, finite topological type, compactification, Euler characteristic.
Supported by the Fundació Caixa Castelló-Bancaixa Grants P1.1B2006-34 and P1.1B2009-14
* Supported by MICINN grant No. MTM2010-21206-C02-02.

Lemma 3.1, and Proposition 3.2) and in the isoperimetric inequality for the extrinsic balls given in Theorem 1.1 in [12]. These results are based, in its turn, on the divergence Theorem and the Hessian and Laplacian comparison theory of restricted distance function, (see [6], [7] and [13]) which involves bounds on the mean curvature of the submanifold.

We have proved the following Chern-Osserman inequality, which encompasses inequalities (1.1) and (1.2):

Theorem A. *Let P^2 be an complete minimal surface immersed in a simply connected real space form with constant sectional curvature $b \leq 0$, $\mathbb{K}^n(b)$. Let us suppose that $\int_P \|B^P\|^2 d\sigma < \infty$. Then*

(1) P has finite topological type.

(2) $\operatorname{Sup}_{t>0}(\frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{b,2})}) < \infty$

(3) $-\chi(P) \leq \frac{\int_P \|B^P\|^2}{4\pi} - \operatorname{Sup}_{t>0} \frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{b,2})}$

where $\chi(P)$ is the Euler characteristic of P.

Although with this approach we are not able to state equality (1.1) in the Euclidean setting, we shall prove in Theorem B the following Chern-Osserman type equality for cmi surfaces in the Hyperbolic space:

Theorem B. *Let P^2 be a complete immersed minimal surface in $\mathbb{H}^n(b)$. Let us suppose that $\int_P \|B^P\|^2 d\sigma < \infty$. Then*

$$-\chi(P) = \frac{1}{4\pi}\int_P \|B^P\|^2 d\sigma - \operatorname{Sup}_{t>0}\frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{b,2})} - \frac{1}{2\pi}G_b(P) \tag{1.3}$$

where $G_b(P)$ is a nonnegative and finite quantity which do not depends on the exhaustion by extrinsic balls $\{D_t\}_{t>0}$ of P and is given by

$$G_b(P) := \lim_{t\to\infty}\left(h_b(t)\operatorname{Vol}(B_t^{b,2})\left(\frac{(\operatorname{Vol}(D_t))}{\operatorname{Vol}(B_t^{b,2})}\right)' + \int_{\partial D_t}\langle B^P(e,e), \frac{\nabla^\perp r}{\|\nabla^P r\|}\rangle d\sigma_t\right) \tag{1.4}$$

1.1. **Outline.** The outline of the paper is following. In Section §.2 we present the basic facts about the Hessian comparison theory of restricted distance function we are going to use, obtaining as a corollary the compactification of cmi surfaces in $\mathbb{K}^n(b)$ with finite total extrinsic curvature, (Corollary 2.3). Section §.3 is devoted to the unified proof of the Chern-Osserman inequality for complete minimal surfaces with finite total extrinsic curvature immersed in Euclidean and Hyperbolic spaces (Theorem A), and in Section §.4 it is proved a Chern-Osserman type equality satisfied by the cmi surfaces in $\mathbb{H}^n(b)$ (Theorem B).

2. Preliminaires

2.1. **The extrinsic distance.** We assume throughout the paper that P^2 is a complete, non-compact, immersed, 2-dimensional submanifold in a simply connected real space form of non-positive constant sectional curvature $\mathbb{K}^n(b)$, ($\mathbb{K}^n(b) = \mathbb{R}^n$ when $b = 0$ and $\mathbb{K}^n(b) = \mathbb{H}^n(b)$ when $b < 0$) . All the points in these manifolds are poles. Recall that a pole is a point o such that the exponential map

$$\exp_o\colon T_oN^n \to N^n$$

is a diffeomorphism. For every $x \in N^n \setminus \{o\}$ we define $r_o(x) = \operatorname{dist}_N(o,x)$, and this distance is realized by the length of a unique geodesic from o to x, which is the *radial geodesic from* o. We also denote by r the restriction $r|_P : P \to \mathbb{R}_+ \cup \{0\}$. This restriction is called the *extrinsic distance function* from o in P^m. The gradients of r in N and P are denoted by $\nabla^N r$ and $\nabla^P r$, respectively. Let us remark that $\nabla^P r(x)$ is just the tangential component in P of $\nabla^N r(x)$, for all $x \in S$. Then we have the following basic relation:

$$\nabla^N r = \nabla^P r + (\nabla^N r)^\perp, \tag{2.1}$$

where $(\nabla^N r)^\perp(x) = \nabla^\perp r(x)$ is perpendicular to T_xP for all $x \in P$.

On the other hand, we should recall that all immersed surfaces P in the real space forms of non-positive constant sectional curvature $N^n = \mathbb{K}^n(b)$ which satisfies $\int_P \|B^P\|^2 d\sigma < \infty$ are properly immersed (see [1], [10] and [11]). Therefore, we can omit the hypothesis about the properness of the immersion when we assume that $\int_P \|B^P\|^2 d\sigma < \infty$.

Definition 2.1. Given a connected and complete surface P^2 properly immersed in a manifold N^n with a pole $o \in N$, we denote the *extrinsic metric balls* of radius $t > 0$ and center $o \in N$ by $D_t(o)$. They are defined as the intersection

$$D_t(o) = B_t^N(o) \cap P = \{x \in P \colon r(x) < t\},$$

where $B_t^N(o)$ denotes the open geodesic ball of radius R centered at the pole o in N^n.

Remark a. We want to point out that the extrinsic domains $D_t(o)$ are precompact sets, (because we assume in the definition above that the submanifold P is properly immersed), with boundary $\partial D_t(o)$ being a immersed curve in P. The generical smoothness of $\partial D_t(o)$ follows from the following considerations: the distance function r is smooth in $\mathbb{K}^n(b)\setminus\{o\}$ since $\mathbb{K}^n(b)$ to possess a pole $o \in \mathbb{K}^n(b)$, ($b \leq 0$). Hence the restriction $r|_P$ is smooth in P and consequently the radii t that produce smooth boundaries $\partial D_t(o)$ are dense in $\mathbb{R}$ by Sard's theorem and the Regular Level Set Theorem.

Remark b. When the submanifold considered is totally geodesic, namely, when P is a Hyperbolic or an Euclidean subespace of the ambient real space form, the extrinsic balls become geodesic balls, and its boundary is the distance sphere. We recall here that the mean curvature of the geodesic sphere in the real space form $\mathbb{K}^n(b)$, 'pointed inward' is (see [12]):

$$h_b(t) = \begin{cases} \sqrt{b}\cot\sqrt{b}t & \text{if } \ b > 0 \\ 1/t & \text{if } \ b = 0 \\ \sqrt{-b}\coth\sqrt{-b}t & \text{if } \ b < 0 \end{cases}$$

2.2. **Hessian comparison analysis of the extrinsic distance.** Let us consider now D_t an extrinsic ball in a complete and properly immersed minimal surface P in the real space form $\mathbb{K}^n(b)$ with $b \leq 0$. We are going to apply Gauss-Bonnet formula to the curve ∂D_t. To do that, we need to compute its geodesic curvature in the following

Proposition 2.2. *Given ∂D_t the smooth closed curves in P,*

$$k_g^{\partial D_t} = \frac{h_b(t)}{\| \nabla^P r\|} + \langle B^P(e,e), \frac{\nabla^\perp r}{\| \nabla^P r\|}\rangle \tag{2.2}$$

Proof. Let $\{e,\nu\} \subset TP$ be an orthonormal frame along the curve ∂D_t, where e is the unit tangent vector to ∂D_t and $\nu = \frac{\nabla^P r}{\| \nabla^P \|}$ is the unit normal to ∂D_t in P, pointed outward.

From the definition of geodesic curvature of the extrinsic boundaries ∂D_t, we have

$$k_g^t = -\langle \nabla_e^P e, \frac{\nabla^P r}{\|\nabla^P r\|}\rangle \tag{2.3}$$

Then, having on account the definition of Hessian

$$Hess^P r(e,e) = \langle \nabla^P \nabla^P r, e\rangle$$

and the fact that $\nabla^P r$ and e are orthogonal,

$$k_g^t = \frac{1}{\|\nabla^P r\|} Hess^P r(e,e) \tag{2.4}$$

But, given $X \in T_qP$ unitary, (see [7] and [13] for detailed computations):

$$\text{Hess}^P(r)(X,X) = h_b(r)\left(1 - \langle X, \nabla^{\mathbb{K}^n(b)} r\rangle^2\right) + \langle \nabla^{\mathbb{K}^n(b)} r, B^P(X,X)\rangle \tag{2.5}$$

where B^P is the second fundamental form of P in N. Applying at this point equation (2.5):

$$k_g^t = \frac{1}{\|\nabla^P r\|}\{h_b(r) + \langle \nabla^\perp r, B^P(e,e)\rangle\} \tag{2.6}$$

□

Now, we consider $\{D_t\}_{t>0}$ an exhaustion of P by extrinsic balls. Recall than an exhaustion of the submanifold P is a sequence of subsets $\{D_t \subseteq P\}_{t>0}$ such that:

- $D_t \subseteq D_s$ when $s \geq t$
- $\cup_{t>0} D_t = P$

Using the equality (2.2) for the geodesic curvature of the extrinsic curves we have the following result

Theorem 2.3. *Let P^2 be an complete minimal surface immersed in a simply connected real space form with constant sectional curvature $b \leq 0$, $\mathbb{K}^n(b)$. Let us suppose that $\int_P \|B^P\|^2 d\sigma < \infty$. Then*

(i) P is diffeomorphic to a compact surface P^* punctured at a finite number of points.

(ii) For all sufficiently large $t > R_0 > 0$, $\chi(P) = \chi(D_t)$ and hence, given $\{D_t\}_{t>0}$ an exhaustion of P by extrinsic balls,

$$\chi(P) = \lim_{t\to\infty} \chi(D_t)$$

Proof. Let us consider $\{D_t\}_{t>0}$ an exhaustion of P by extrinsic balls, centered at the pole $o \in \mathbb{K}^n(b)$. We apply Lemma 2.2 to the smooth curves ∂D_t: As

$$-\|B^P\| \leq \langle B^P(e,e), \nabla^\perp r\rangle \leq \|B^P\|$$

we have, on the points of the curve $q \in \partial D_t$,

$$\begin{aligned} \| \nabla^P r\|(q) \cdot k_g^{\partial D_t}(q) &= h_b(r_o(q)) + \langle B^P(e,e), \nabla^\perp r\rangle(q) \\ &\geq h_b(r_o(q)) - \|B^P\|(q) \end{aligned} \tag{2.7}$$

Using now Proposition 2.2 in [1], when P^2 is a cmi in $\mathbb{R}^n$ or Lemma 3.1 in [11], when P^2 is a cmi in $\mathbb{H}^n(b)$, we know that $\|B^P\|(q)$ goes uniformly to 0 as $t = r_o(q) \to \infty$. Hence, for all the points $q \in \partial D_t$ and for sufficiently large t,

$$\| \nabla^P r\|(q) \cdot k_g^{\partial D_t}(q) > 0 \tag{2.8}$$

Hence, $\| \nabla^P r\| > 0$ in ∂D_t, for all sufficiently large t. Fixing a sufficienty large radius R_0, we can conclude that the extrinsic distance r_o has no critical points in $P \setminus \mathrm{D}_{R_0}$.

The above inequality implies that for this sufficienty large fixed radius R_0, there is a diffeomorphism

$$\Phi : P \setminus \mathrm{D}_{R_0} \to \partial D_{R_0} \times [0,\infty[$$

In particular, P has only finitely many ends, each of finite topological type.

To proof this we apply Theorem 3.1 in [9], concluding that, as the extrinsic annuli $A_{R_0,R}(o) = D_R(o) \setminus D_{R_0}(o)$ contains no critical points of the extrinsic distance function $r_o : P \longrightarrow \mathbb{R}^+$ because inequality (2.8), then $D_R(o)$ is diffeomorphic to $D_{R_0}(o)$ for all $R \geq R_0$.

The above diffeomorfism implies that we can construct P from D_{R_0} (R_0 big enough) attaching annulis and that $\chi(P \setminus D_t) = 0$ when $t \geq R_0$. Then, for all $t > R_0$,

$$\chi(P) = \chi(D_t \cup (P \setminus D_t)) = \chi(D_t)$$

□

3. Proof of Theorem A

We begin with the following results which are the common ingredient of the proof, both for the Euclidean and Hyperbolic cases :

Lemma 3.1. *Let $P^2 \subset \mathbb{K}^n(b)$ be a surface properly immersed in a real space form with curvature $b \leq 0$, let D_t be an extrinsic disc in P of radius $t > 0$ and let ∂D_t the extrinsic circle. Then:*

$$\int_{\partial D_t} \frac{||\nabla^\perp r||^2}{||\nabla^P r||} d\sigma_t \leq \int_{\partial D_t} \frac{1}{||\nabla^P r||} - h_b(t)\operatorname{Vol}(D_t) d\sigma_t \tag{3.1}$$

Proof. Tracing equality (2.5) we obtain the following expression for the Laplacian of the extrinsic distance in this context:

$$\Delta^P(r) \,=\, (m - \|\nabla^P r\|^2) h_b(r) + m\langle \nabla^N r,\, H_P \rangle \quad , \tag{3.2}$$

where H_P denotes the mean curvature vector of P in N and $h_b(r)$ is the mean curvature of the geodesic r-spheres in $\mathbb{K}^n(b)$. Applying divergence theorem we have

$$\begin{aligned}
&\int_{\partial D_t} \frac{||\nabla^\perp r||^2}{||\nabla^P r||} d\sigma_t = \int_{\partial D_t} \frac{1}{||\nabla^P r||} d\sigma_t - \int_{\partial D_t} ||\nabla^P r|| d\sigma_t = \int_{\partial D_t} \frac{1}{||\nabla^P r||} d\sigma_t \\
&- \int_{D_t} \Delta^P r d\sigma = \int_{\partial D_t} \frac{1}{||\nabla^P r||} d\sigma_t - \int_{D_t} (2 - ||\nabla^P r||^2) h_b(r) d\sigma \\
&\leq \int_{\partial D_t} \frac{1}{||\nabla^P r||} d\sigma_t - \int_{D_t} h_b(r) d\sigma \leq \int_{\partial D_t} \frac{1}{||\nabla^P r||} d\sigma_t - h_b(t)\operatorname{Vol}(D_t)
\end{aligned} \tag{3.3}$$

□

Proposition 3.2. *Let $P^2 \subset \mathbb{K}^n(b)$ be a complete minimal surface properly immersed in a real space form with curvature $b \leq 0$, let D_t be an extrinsic disc in P of radius $t > 0$ and let ∂D_t be its boundary. Then:*

$$\begin{aligned}
&-2\pi\chi(D_t) + (b + \frac{f_{b,\alpha}^2(t) h_b(t)}{2}) \operatorname{Vol}(D_t) \\
&\quad + (h_b(t) - \frac{f_{b,\alpha}^2(t)}{2}) \int_{\partial D_t} \frac{1}{\|\nabla^P r\|} d\sigma_t \leq \frac{1}{2} R(t) + \frac{1}{2 f_{b,\alpha}^2(t)} R'(t)
\end{aligned} \tag{3.4}$$

where $R(t) = \int_{D_t} \|B^P\|^2 d\sigma$, $\|B^P\|$ is the norm of the second fundamental form of P in $\mathbb{K}^n(b)$, $\chi(D_t)$ is the Euler's characterisc of D_t and, given $\alpha \in]0, 2[$,

$$f_{b,\alpha}^2(t) = \alpha h_b(t)$$

Proof. Integrating along ∂D_t equation (2.2) and using Gauss-Bonnet theorem and co-area formula, (see [14]), we obtain

$$\begin{aligned}
&2\pi\chi(D_t) - \int_{D_t} K^P d\sigma = \\
&h_b(t) \int_{\partial D_t} \frac{1}{\| \nabla^P r\|} d\sigma_t + \int_{\partial D_t} \langle B^P(e, e), \frac{\nabla^\perp r}{\| \nabla^P r\|} \rangle d\sigma_t
\end{aligned} \tag{3.5}$$

where we denote as K^P the Gauss curvature of P.

But , on ∂D_t,

$$-\|B^P\|\frac{\| \nabla^\perp r\|}{\| \nabla^P r\|} \leq \langle B^P(e,e), \frac{\nabla^\perp r}{\| \nabla^P r\|}\rangle \leq \|B^P\|\frac{\| \nabla^\perp r\|}{\| \nabla^P r\|}$$

so, as $f_{b,\alpha}(t) \geq 0 \,\forall t > 0$, having into account the inequality among the arithmetic and geometric mean and applying co-area formula:

$$\begin{aligned}
2\pi\chi(D_t) - \int_{D_t} K^P d\sigma &= h_b(t)\int_{\partial D_t} \frac{1}{\| \nabla^P r\|} d\sigma_t \\
&+ \int_{\partial D_t} \langle B^P(e,e), \frac{\nabla^\perp r}{\| \nabla^P r\|}\rangle d\sigma_t \;\geq\; h_b(t)\int_{\partial D_t} \frac{1}{\| \nabla^P r\|} d\sigma_t \\
&- \frac{1}{2}\int_{\partial D_t} \frac{\|B^P\|^2}{f^2_{b,\alpha}(r)\| \nabla^P r\|} d\sigma_t - \frac{1}{2}\int_{\partial D_t} \frac{f^2_{b,\alpha}(r)\| \nabla^\perp r\|^2}{\| \nabla^P r\|} d\sigma_t \\
&\geq h_b(t)\int_{\partial D_t} \frac{1}{\| \nabla^P r\|} d\sigma_t - \frac{1}{2f^2_{b,\alpha}(t)}R'(t) - \frac{f^2_{b,\alpha}(t)}{2}\int_{\partial D_t} \frac{\| \nabla^\perp r\|^2}{\| \nabla^P r\|} d\sigma_t
\end{aligned} \tag{3.6}$$

Then, using inequality (3.1) of Lemma 3.1 in the last member of the inequalities (3.6) and applying Gauss equation for minimal surfaces in the real space forms $\mathbb{K}^n(b)$, we have

$$\begin{aligned}
2\pi\chi(D_t) - b\,\mathrm{Vol}(D_t) + \frac{1}{2}R(t) &\geq (h_b(t) - \frac{f^2_{b,\alpha}(t)}{2})\int_{\partial D_t} \frac{1}{\| \nabla^P r\|} d\sigma_t \\
&- \frac{1}{2f^2_{b,\alpha}(t)}R'(t) + \frac{f^2_{b,\alpha}(t)h_b(t)}{2}\,\mathrm{Vol}(D_t)
\end{aligned} \tag{3.7}$$

and hence

$$\begin{aligned}
-2\pi\chi(D_t) &+ (b + \frac{f^2_{b,\alpha}(t)h_b(t)}{2})\,\mathrm{Vol}(D_t) \\
&+ (h_b(t) - \frac{f^2_{b,\alpha}(t)}{2})\int_{\partial D_t} \frac{1}{\|\nabla^P r\|} \leq \frac{1}{2}R(t) + \frac{1}{2f^2_{b,\alpha}(t)}R'(t)
\end{aligned} \tag{3.8}$$

□

We are going to divide the proof in two cases: the *Case I*, where the ambient space is the Hyperbolic space $\mathbb{H}^n(b)$, and the *Case II* where the ambient space is the Euclidean space $\mathbb{R}^n$.

Case I. Let us consider P (properly) immersed in $\mathbb{H}^n(b)$. Let $\{D_t\}_{t>0}$ be an exhaustion of P by extrinsic balls. Using co-area formula, we know that

$$\frac{d}{dt}\,\mathrm{Vol}(D_t) = \int_{\partial D_t} \frac{1}{\|\nabla^P r\|} d\sigma_t \tag{3.9}$$

Hence, applying Proposition 3.2 we have

$$\begin{aligned}
-2\pi\chi(D_t) &+ (b + \frac{f^2_{b,\alpha}(t)h_b(t)}{2})\,\mathrm{Vol}(D_t) \\
&+ (h_b(t) - \frac{f^2_{b,\alpha}(t)}{2})\frac{d}{dt}\,\mathrm{Vol}(D_t) \leq \frac{1}{2}R(t) + \frac{1}{2f^2_{b,\alpha}(t)}R'(t)
\end{aligned} \tag{3.10}$$

On the other hand, from 3.9, $\frac{d}{dt}\,\mathrm{Vol}(D_t) \geq \mathrm{Vol}(\partial D_t)$. Therefore, using inequality (3.10) we obtain

$$\begin{aligned}
&- 2\pi\chi(D_t)\\
&+ \operatorname{Vol}(D_t)\left[(b+\frac{f_{b,\alpha}^2(t)h_b(t)}{2}) + (h_b(t) - \frac{f_{b,\alpha}^2(t)}{2})\frac{\operatorname{Vol}(\partial D_t)}{\operatorname{Vol}(D_t)}\right]\\
&\leq \frac{1}{2}R(t) + \frac{1}{2f_{b,\alpha}^2(t)}R'(t)
\end{aligned} \tag{3.11}$$

Applying isoperimetric inequality in [12], (Theorem 1.1), we have

$$\begin{aligned}
&- 2\pi\chi(D_t)\\
&+ \operatorname{Vol}(D_t)\left[(b+\frac{f_{b,\alpha}^2(t)h_b(t)}{2}) + (h_b(t) - \frac{f_{b,\alpha}^2(t)}{2})\frac{\operatorname{Vol}(S_t^{b,1})}{\operatorname{Vol}(B_t^{b,2})}\right]\\
&\leq \frac{1}{2}R(t) + \frac{1}{2f_{b,\alpha}^2(t)}R'(t)
\end{aligned} \tag{3.12}$$

Hence, using the fact that

$$b\operatorname{Vol}(B_t^{b,2}) + h_b(t)\operatorname{Vol}(S_t^{b,1} = 2\pi \;\; \forall t > 0$$

we obtain, with some computations

$$\begin{aligned}
&-2\pi\chi(D_t) + \frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{b,2})}\left[2\pi - 2\pi\frac{f_{b,\alpha}^2(t)}{2}\frac{\operatorname{Vol}(B_t^{b,2})}{\operatorname{Vol}(S_t^{b,1})}\right]\\
&\leq \frac{1}{2}R(t) + \frac{1}{2f_{b,\alpha}^2(t)}R'(t)
\end{aligned} \tag{3.13}$$

Therefore, for all $t > 0$,

$$\begin{aligned}
&\frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{b,2})}\left(1 - \frac{\alpha h_b(t)}{2}\frac{\operatorname{Vol}(B_t^{b,2})}{\operatorname{Vol}(S_t^{b,1})}\right) - \chi(D_t)\\
&\qquad\qquad \leq \frac{R(t)}{4\pi} + \frac{R'(t)}{4\pi\alpha h_b(t)}
\end{aligned} \tag{3.14}$$

As $\frac{||B^P||^2}{h_b(t)} \leq \frac{1}{\sqrt{-b}}||B^P||^2$, then $\int_P ||B^P||^2 d\sigma < \infty$ implies $\int_P \frac{||B^P||^2}{h_b(t)} d\sigma < \infty$. Hence, by co-area formula:

$$\int_0^\infty \left(\int_{\partial D_t} \frac{||B^P||^2}{||\nabla^P r|| h_b(r)}\right) dt = \int_0^\infty \left(\frac{R'(t)}{h_b(t)}\right) dt < \infty \tag{3.15}$$

Therefore, there is a monotone increasing (sub)sequence $\{t_i\}_{i=1}^\infty$ tending to infinity, (namely, $t_i \to \infty$ when $i \to \infty$), such that $\frac{R'(t_i)}{h_b(t_i)} \to 0$ when $i \to \infty$.

Let us consider the exhaustion of P by these extrinsic balls, namely, $\{D_{t_i}\}_{i=1}^\infty$. Then we have, replacing t for t_i and taking limits when $i \to \infty$ in inequality (3.14) and applying Theorem 2.3 (ii),

$$\begin{aligned}
&\operatorname{Sup}_i \frac{\operatorname{Vol}(D_{t_i})}{\operatorname{Vol}(B_{t_i}^{b,2})}\left(1 - \frac{\alpha}{2}\right) - \chi(P)\\
&\qquad \leq \lim_{i\to\infty} \frac{R(t_i)}{4\pi} = \frac{1}{4\pi}\int_P \|B^P\|^2 d\sigma < \infty
\end{aligned} \tag{3.16}$$

for all α such that $0 < \alpha < 2$.

Hence, as $\frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{b,2})}$ is a continuous non decreasing function of t, we can conclude that $\operatorname{Sup}_{t>0} \frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{b,2})} < \infty$ and $-\chi(P) < \infty$.

Then, letting α tend to 0 in (3.16), we get, for all $t > 0$:

$$\operatorname{Sup}_{t>0} \frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{b,2})} - \chi(P) \leq \frac{\int_P \|B^P\|^2}{4\pi} \tag{3.17}$$

Case II. Let us consider P immersed in $\mathbb{R}^n$. We consider, as in the proof above, an exhaustion of P by extrinsic balls, $\{D_t\}_{t>0}$, but now, and following [1], these extrinsic balls will be centered at the origin $0 \in \mathbb{R}^n$, which we assume, without loss of generality, that belongs to the surface P. Applying Proposition 3.2 we have

$$\begin{aligned} &-2\pi\chi(D_t) + (\frac{\alpha}{2t^2}) \operatorname{Vol}(D_t) \\ &\quad + (\frac{1}{t} - \frac{\alpha}{2t}) \int_{\partial D_t} \frac{1}{\|\nabla^P r\|} \leq \frac{1}{2} R(t) + \frac{t}{2\alpha} R'(t) \end{aligned} \tag{3.18}$$

Now, as $\int_P ||B^P||^2 d\sigma < \infty$, we can apply Proposition 2.2 in [1], so we have, for $\alpha \in]0, 2[$,

$$\frac{t}{2\alpha} R'(t) = \frac{t}{2\alpha} \int_{\partial D_t} \frac{\|B^P\|^2}{\|\nabla^P r\|} d\sigma \leq \frac{\mu(t)}{2\alpha t} \int_{\partial D_t} \frac{1}{\|\nabla^P r\|} d\sigma \tag{3.19}$$

being $\mu(t)$ such that $\lim_{t\to\infty} \mu(t) = 0$ and therefore, from (3.18),

$$\begin{aligned} &-2\pi\chi(D_t) + \operatorname{Vol}(D_t)(\frac{\alpha}{2t^2}) \\ &+ (\frac{1}{t} - \frac{\alpha}{2t} - \frac{\mu(t)}{2\alpha t}) \int_{\partial D_t} \frac{1}{\|\nabla^P r\|} d\sigma_t \;\leq\; \frac{1}{2} R(t) \end{aligned} \tag{3.20}$$

On the other hand, $\frac{1}{t} - \frac{\alpha}{2t} - \frac{\mu(t)}{2\alpha t} \geq 0$ if and only if $\mu(t) \leq \alpha(2-\alpha)$, which it is true for t big enough, namely, for $t > t_\alpha$ because $\lim_{t\to\infty} \mu(t) = 0$. Hence, as $\operatorname{Vol}(\partial D_t) \leq \int_{\partial D_t} \frac{1}{\|\nabla^P r\|} d\sigma_t$, and applying Theorem 1.1 in [12], we have that inequality (3.20) becomes, for all $t > t_\alpha$

$$\begin{aligned} &-2\pi\chi(D_t) \\ &+ \frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{0,2})} \left[2\pi(1 - \frac{\alpha}{2} - \frac{\mu(t)}{2\alpha}) + \frac{\pi\alpha}{2}\right] \leq \frac{1}{2} R(t) \end{aligned} \tag{3.21}$$

Then, taking limits when $t \to \infty$ in inequality (3.21) and applying Theorem 2.3, we have that $\lim_{t\to\infty} \mu(t) = 0$ and $\chi(P) = \lim_{t\to\infty} \chi(D_t)$, so we obtain, for all α such that $0 < \alpha < 2$:

$$\begin{aligned} &2\pi \operatorname{Sup}_t \frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{0,2})} \left(1 - \frac{\alpha}{2} + \frac{\pi\alpha}{2}\right) \\ &\qquad\qquad - 2\pi\chi(P) \leq \frac{\int_P \|B^P\|^2}{2} < \infty \end{aligned} \tag{3.22}$$

Therefore we obtain $\operatorname{Sup}_{t>0} \frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{0,2})} < \infty$ and $-\chi(P) < \infty$.

Then, letting α tend to 0 we obtain, for all $t > 0$:

$$\operatorname{Sup}_{t>0} \frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{0,2})} - \chi(P) \leq \frac{\int_P \|B^P\|^2}{4\pi} \tag{3.23}$$

4. PROOF OF THEOREM B

In Corollary 2.3, it was obtained a sufficienty large radius R_0, such that the extrinsic distance r_p has no critical points in $P \setminus \mathrm{D}_{R_0}$.

Hence for this sufficienty large fixed radius R_0, there is a diffeomorphism

$$\Phi : P \setminus \mathrm{D}_{R_0} \to \partial D_{R_0} \times [0, \infty[$$

so, in particular, P has only finitely many ends, each of finite topological type.

The above diffeomorfism implied that we could construct P from D_{R_0} (R_0 big enough) attaching annulis and that $\chi(P \setminus D_t) = 0$ when $t \geq R_0$, and hence for all $t > R_0$, $\chi(P) = \chi(D_t)$.

Let us consider now an exhaustion by extrinsic balls $\{D_t\}_{t>0}$ of P such that the extrinsic distance r_o has no critical points in $P \setminus \mathrm{D}_{R_0}$.

Applying now Gauss-Bonnet Theorem to the extrinsic balls D_t

$$2\pi\chi(P) = \int_{D_t} K^P d\sigma + \int_{\partial D_t} k_g d\sigma_t \tag{4.1}$$

Having in to account equation (2.2) and the Gauss formula, we have, for all sufficiently large radius $t > R_0$

$$\begin{aligned} 2\pi\chi(P) = &-\frac{1}{2}\int_{D_t} \|B^P\|^2 + b\,\mathrm{Vol}(D_t) + h_b(t)\,(\mathrm{Vol}(D_t))' \\ &+ \int_{\partial D_t} \langle B^P(e,e), \frac{\nabla^\perp r}{\|\nabla^P r\|}\rangle d\sigma_t = -\frac{1}{2}\int_{D_t} \|B^P\|^2 d\sigma \\ &+ \frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B_t^{b,2})}\left(b\cdot \mathrm{Vol}(B_t^{b,2}) + h_b(t)(\mathrm{Vol}(D_t))' \frac{\mathrm{Vol}(B_t^{b,2})}{\mathrm{Vol}(D_t)} \right. \\ &\left. + \frac{\mathrm{Vol}(B_t^{b,2})}{\mathrm{Vol}(D_t)} \int_{\partial D_t} \langle B^P(e,e), \frac{\nabla^\perp r}{\|\nabla^P r\|}\rangle d\sigma_t \right) \end{aligned} \tag{4.2}$$

But $2\pi = b\cdot \mathrm{Vol}(B_t^{b,2}) + h_b(t)\,\mathrm{Vol}(S_t^{b,1})\ \ \forall t > 0$, so, for all sufficiently large radius $t > R_0$ and after some computations:

$$\begin{aligned} 2\pi\chi(P) = &-\frac{1}{2}\int_{D_t} \|B^P\|^2 d\sigma + 2\pi \frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B_t^{b,2})} + h_b(t)\,\mathrm{Vol}(B_t^{b,2})(\frac{(\mathrm{Vol}(D_t))}{\mathrm{Vol}(B_t^{b,2})})' \\ &+ \int_{\partial D_t} < B^P(e,e), \frac{\nabla^\perp r}{\|\nabla^P r\|} > d\sigma_t \end{aligned} \tag{4.3}$$

The above equation is valid for all $t > R_0$, so, taking limits when $t \to \infty$, we can define

$$\begin{aligned} G_b(P) := \lim_{t\to\infty} &\left(h_b(t)\,\mathrm{Vol}(B_t^{b,2})(\frac{(\mathrm{Vol}(D_t))}{\mathrm{Vol}(B_t^{b,2})})' \right. \\ &\left. + \int_{\partial D_t} < B^P(e,e), \frac{\nabla^\perp r}{\|\nabla^P r\|} > d\sigma_t) \right) \end{aligned} \tag{4.4}$$

Using equalities (4.3), we have that

$$G_b(P) = 2\pi\chi(P) + \frac{1}{2}\int_{D_t} \|B^P\|^2 d\sigma - 2\pi\, \mathrm{Sup}_t \frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B_t^{b,2})} < \infty \tag{4.5}$$

and hence, $G_b(P)$ do not depends on the exhaustion $\{D_t\}_{t>0}$.

REFERENCES

[1] M. T. Anderson, *The compactification of a minimal submanifold in Euclidean space by the Gauss map* , I.H.E.S. Preprint, 1984

[2] S.S. Chern and R. Osserman *Complete minimal surfaces in euclidean space* (Academic Press, New York, 1984).

[3] S.S. Chern and R. Osserman, *Complete minimal surface in* E^n, J. d'Analyse Math. **19** (1967), 15-34.

[4] Chen Qing, *On the area growth of minimal surfaces in* $\mathbb{H}^n$, Geometriae Dedicata, **75** (1999), 263–273.

[5] Chen Qing and Cheng Yi, *Chern-osserman inequality for minimal surfaces in* $\mathbb{H}^n$, Proc. Amer. Math Soc., Vol. 128, **8**, (1999), 2445-2450.

[6] R. Greene and S. Wu *Function theory on manifolds which posses a pole*, Lecture Notes in Math.,**699**, (1979), Springer Verlag, Berlin.

[7] A. Hurtado and V. Palmer, *A note on the p-parabolicity of submanifolds*, Pot. Analysis **34**, (2), (2011),101–118.

[8] L.P. Jorge and W. H. Meeks, *The topology of minimal surfaces of finite total Gaussian curvature*, Topology, **122**, (1983), 203-221.

[9] J. Milnor, *Morse theory*, Princeton University Press, New Jersey, 1969.

[10] S. Muller and V. Sverak, *On surfaces of finite total curvature*, J. Differential Geometry, **42**, 2, (1995), 229-257.

[11] G. De Oliveira, *Compactification of minimal submanifolds of hyperbolic space*, Comm. An. and Geom., **1** (1993), 1-29.

[12] V. Palmer, *Isoperimetric Inequalities for extrinsic balls in minimal submanifolds and their applications*, J. London Math. Soc. (2) **60** (1999), 607-616.

[13] V. Palmer, *On deciding whether a submanifold is parabolic of hyperbolic using its mean curvature* Simon Stevin Transactions on Geometry, vol 1. 131-159, Simon Stevin Institute for Geometry, Tilburg, The netherlands, 2010.

[14] T. Sakai, *Riemannian Geometry*, Translations of Mathematical Monographs, vol. 149, A.M.S.1996.

DEPARTAMENT DE MATEMÀTIQUES-INIT, UNIVERSITAT JAUME I, CASTELLÓ, SPAIN.
E-mail address: gimenov@guest.uji.es

DEPARTAMENT DE MATEMÀTIQUES-INIT, UNIVERSITAT JAUME I, CASTELLÓ, SPAIN.
E-mail address: palmer@mat.uji.es

4

Creixement del volum de subvarietats i constant isoperimètrica de Cheeger

- Article publicat en Arxiv.org com *arXiv:1104.5625v2 [math.DG]*.
- Article acceptat per a la seua publicació en *Proceedings of the American Mathematical Society*.

VOLUME GROWTH OF SUBMANIFOLDS AND THE CHEEGER ISOPERIMETRIC CONSTANT

VICENT GIMENO# AND VICENTE PALMER#

ABSTRACT. We obtain an estimate of the Cheeger isoperimetric constant in terms of the volume growth for a properly immersed submanifold in a Riemannian manifold which possesses at least one pole and sectional curvature bounded from above .

1. INTRODUCTION

The Cheeger isoperimetric constant $\mathcal{I}_\infty(M)$ (see [5]) of a non-compact Riemannian manifold of dimension $n \geq 2$ is defined as:

$$\mathcal{I}_\infty(M) := \inf_{\Omega} \left\{ \frac{\mathrm{Vol}(\partial\Omega)}{\mathrm{Vol}(\Omega)} \right\} \tag{1.1}$$

where Ω ranges over open submanifolds of M possessing compact closure and smooth boundary, $\mathrm{Vol}(\partial\Omega)$ denotes the $(n-1)$-dimensional volume of the boundary $\partial\Omega$, and $\mathrm{Vol}(\Omega)$ denotes the n-dimensional volume of Ω, (concerning this definition, see also [3] and [4]).

This paper focuses on obtaining sharp upper and lower bounds for the Cheeger isoperimetric constant $\mathcal{I}_\infty(P)$ of a complete submanifold P with controlled mean curvature and properly immersed in an ambient manifold N with sectional curvatures bounded from above and which possess at least one pole.

As a consequence of these upper and lower bounds, and as a preliminary view of our main theorems (Theorems 3.2 and 3.3 in section §.3), we present the following results, which constitute a particular case of them when a complete, non-compact and minimal submanifold properly immersed in a Cartan-Hadamard manifold is considered. In contrast, if we focus on compact and minimal submanifolds of a Riemannian manifold satisfying other geometric restrictions, we refer to the work [12], where certain isoperimetric inequalities involving these submanifolds have been proven.

Theorem A. *Let P^m be a complete non-compact and minimal submanifold properly immersed in a Cartan-Hadamard manifold N with sectional curvatures bounded from above as $K_N \leq b \leq 0$, and suppose that $\mathrm{Sup}_{t>0}(\frac{\mathrm{Vol}(P\cap B_t^N)}{\mathrm{Vol}(B_t^{m,b})}) < \infty$, where B_t^N is the geodesic t-ball in the ambient manifold N and $B_t^{m,b}$ denotes the geodesic t-ball in the real space form of constant sectional curvature $\mathbb{K}^m(b)$.*

Then

$$\mathcal{I}_\infty(P) \leq (m-1)\sqrt{-b} \quad . \tag{1.2}$$

Theorem B. *Let P^m be a complete non-compact and minimal submanifold properly immersed in a Cartan-Hadamard manifold N with sectional curvatures bounded from above*

2000 *Mathematics Subject Classification.* Primary 53C20, 53C42.

Key words and phrases. Cheeger isoperimetric constant, volume growth, submanifold, Chern-Osserman inequality.

Supported by Fundació Caixa Castelló-Bancaixa Grants P1.1B2006-34 and P1.1B2009-14 and by MICINN grant No. MTM2010-21206-C02-02.

as $K_N \leq b \leq 0$. *Then*

$$\mathcal{I}_\infty(P) \geq (m-1)\sqrt{-b} \quad . \tag{1.3}$$

The lower bounds for $\mathcal{I}_\infty(P)$ in Theorem B come from direct application of the divergence theorem to the Laplacian of the extrinsic distance defined on the submanifold using the distance in the ambient manifold, following the arguments of Proposition 3 in [21] and of Theorem 6.4 in [4].

On the other hand, the upper bounds in Theorem A were obtained by assuming that the (extrinsic) volume growth of the submanifold is bounded from above by a finite quantity. As we shall see in the corollaries, when the submanifold is a minimal immersion in the Euclidean space or when we are dealing with minimal surfaces in the Euclidean or the Hyperbolic space, this crucial fact relates Cheeger's constant $\mathcal{I}_\infty(P)$ with the total extrinsic curvature of the submanifold $\int_P \|B^P\|^m d\sigma$, in the sense that the finiteness of this total extrinsic curvature implies the upper bounds for Cheeger's constant, using the results in [1], [6] and [8].

These lower and upper bounds of $\mathcal{I}_\infty(P)$ given in Theorems 3.2 and 3.3 come from comparisons for the Laplacian of the extrinsic distance defined on the submanifold, and the techniques used to obtain these comparisons are based on the Hessian analysis of this restricted distance function. When the extrinsic curvature of the submanifold is bounded (from above or from below), this analysis focuses on the relation, given in [10], between the Hessian of this function and these (extrinsic) curvature bounds, thus providing comparison results for the Hessian and the Laplacian of the distance function in the submanifold.

The model used in these comparisons is constructed from the corresponding values for these operators computed for the intrinsic distance of a rotationally symmetric space whose sectional curvatures bound the corresponding curvatures of the ambient manifold.

We shall see that the Cheeger constant $\mathcal{I}_\infty(P)$ is bounded by the limit of some isoperimetric quotient determined by the geodesic r-balls in these model spaces, which involves the mean curvature of the submanifold.

1.1. **Outline of the paper.** In section §.2 we present the basic definitions and facts concerning the extrinsic distance restricted to a submanifold, and about the rotationally symmetric spaces used as a model for comparison. We also present the basic results regarding the Hessian comparison theory of restricted distance function that will be used. This section finishes with the description of the isoperimetric context where the results hold. Section §.3 is devoted to the statement and proof of the two main Theorems 3.2 and 3.3 and three corollaries are stated and proven in the final section §.4.

2. Preliminaires

2.1. **The extrinsic distance.** We assume throughout the paper that P^m is a complete, non-compact, properly immersed, m-dimensional submanifold in a complete Riemannian manifold N^n which possesses at least one pole $o \in N$. Recall that a pole is a point o such that the exponential map

$$\exp_o \colon T_o N^n \to N^n$$

is a diffeomorphism. For every $x \in N^n \setminus \{o\}$ we define $r(x) = r_o(x) = \operatorname{dist}_N(o, x)$, and this distance is realized by the length of a unique geodesic from o to x, which is the *radial geodesic from* o. We also denote by r the restriction $r|_P : P \to \mathbb{R}_+ \cup \{0\}$. This restriction is called the *extrinsic distance function* from o in P^m. The gradients of r in N and P are denoted by $\nabla^N r$ and $\nabla^P r$, respectively. Let us remark that $\nabla^P r(x)$ is just the tangential component in P of $\nabla^N r(x)$, for all $x \in S$. Then we have the following basic relation:

$$\nabla^N r = \nabla^P r + (\nabla^N r)^\perp \tag{2.1}$$

where $(\nabla^N r)^\perp(x) = \nabla^\perp r(x)$ is perpendicular to $T_x P$ for all $x \in P$.

Definition 2.1. Given a connected and complete submanifold P^m properly immersed in a manifold N^n with a pole $o \in N$, we denote the *extrinsic metric balls* of radius $t > 0$ and center $o \in N$ by $D_t(o)$. They are defined as the intersection

$$B_t^N(o) \cap P = \{x \in P \colon r(x) < t\}$$

where $B_t^N(o)$ denotes the open geodesic ball of radius t centered at the pole o in N^n.

Remark a. The extrinsic domains $D_t(o)$ are precompact sets (because in the definition above it was assumed that the submanifold P is properly immersed), with smooth boundary $\partial D_t(o)$. The assumption on the smoothness of $\partial D_t(o)$ makes no restriction. Indeed, the distance function r is smooth in $N \setminus \{o\}$ since N is assumed to possess a pole $o \in N$. Hence the restriction $r|_P$ is smooth in P and consequently the radii t that produce smooth boundaries $\partial D_t(o)$ are dense in $\mathbb{R}$ by Sard's theorem and the Regular Level Set Theorem.

We now present the curvature restrictions which constitute the geometric framework of our study.

Definition 2.2. Let o be a point in a Riemannian manifold N and let $x \in N - \{o\}$. The sectional curvature $K_N(\sigma_x)$ of the two-plane $\sigma_x \in T_xN$ is then called a *o-radial sectional curvature* of N at x if σ_x contains the tangent vector to a minimal geodesic from o to x. We denote these curvatures by $K_{o,N}(\sigma_x)$.

In order to control the mean curvatures $H_P(x)$ of P^m at distance r from o in N^n we introduce the following definition:

Definition 2.3. The o-radial mean curvature function for P in N is defined in terms of the inner product of H_P with the N-gradient of the distance function $r(x)$ as follows:

$$\mathcal{C}(x) = -\langle \nabla^N r(x), H_P(x)\rangle \quad \text{for all} \quad x \in P \ .$$

2.2. **Model Spaces.** The model spaces M_w^m are rotationally symmetric spaces which serve as comparison controllers for the radial sectional curvatures of the ambient space N^n.

Definition 2.4 (see [11], [10])**.** A $w-$model M_w^m is a smooth warped product with base $B^1 = [\,0,\, R[\, \subset \mathbb{R}$ (where $0 < R \leq \infty$), fiber $F^{m-1} = S_1^{m-1}$ (i.e., the unit $(m-1)-$sphere with standard metric), and warping function $w \,:\, [\,0,\, R[\to \mathbb{R}_+ \cup \{0\}$ with $w(0) = 0$, $w'(0) = 1$, and $w(r) > 0$ for all $r > 0$. The point $o_w = \pi^{-1}(0)$, where π denotes the projection onto B^1, is called the *center point* of the model space. If $R = \infty$, then o_w is a pole of M_w^m.

Remark b. The simply connected space forms $\mathbb{K}^m(b)$ of constant curvature b can be constructed as $w-$models $\mathbb{K}^n(b) = M_{w_b}^n$ with any given point as the center point using the warping functions

$$w_b(r) = \begin{cases} \frac{1}{\sqrt{b}}\sin(\sqrt{b}\,r) & \text{if } b > 0 \\ \quad r & \text{if } b = 0 \\ \frac{1}{\sqrt{-b}}\sinh(\sqrt{-b}\,r) & \text{if } b < 0 \quad . \end{cases} \tag{2.2}$$

Note that for $b > 0$ the function $w_b(r)$ admits a smooth extension to $r = \pi/\sqrt{b}$. For $b \leq 0$ any center point is a pole.

Remark c. The sectional curvatures of the model spaces K_{o_w, M_w} in the radial directions from the center point are determined by the radial function $K_{o_w, M_w}(\sigma_x) \;=\; K_w(r) \;=\; -\dfrac{w''(r)}{w(r)}$, (see [10], [11] [18]). Moreover, the mean curvature of the distance sphere of radius r from the center point is

$$\eta_w(r) = \frac{w'(r)}{w(r)} = \frac{d}{dr}\ln(w(r)) \quad . \tag{2.3}$$

Hence, the sectional curvature of $\mathbb{K}^n(b)$ is given by $-\dfrac{w_b''(r)}{w_b(r)} = b$ and the mean curvature of the geodesic $r-$sphere $S_r^{w_b} = S_r^{b,n-1}$ in the real space form $\mathbb{K}^n(b)$, 'pointed inward' is (see [19]):

$$\eta_{w_b} = h_b(t) = \begin{cases} \sqrt{b}\cot\sqrt{b}t & \text{if } b > 0 \\ 1/t & \text{if } b = 0 \\ \sqrt{-b}\coth\sqrt{-b}t & \text{if } b < 0 \end{cases} .$$

In particular, in [16] we introduced, for any given warping function $w(r)$, the isoperimetric quotient function $q_w(r)$ for the corresponding $w-$model space M_w^m as follows:

$$q_w(r) = \frac{\operatorname{Vol}(B_r^w)}{\operatorname{Vol}(S_r^w)} = \frac{\int_0^r w^{m-1}(t)\,dt}{w^{m-1}(r)} . \tag{2.4}$$

where B_r^w and S_r^w denotes the metric $r-$ball and the metric $r-$sphere in M_w^m respectively.

2.3. **Hessian comparison analysis of the extrinsic distance.** This subsection offers a corollary of the Hessian comparison Theorem A in [10], which concerns the bounds for the Laplacian of a radial function defined on the submanifold (see [13] and [20] for detailed computations, see also [14]).

Theorem 2.5. *Let N^n be a manifold with a pole o and let M_w^m denote a $w-$model with center o_w. Let P^m be a properly immersed submanifold in N. Then we have the following dual Laplacian inequalities for modified distance functions $f \circ r : P \longrightarrow \mathbb{R}$:*

Suppose that every o-radial sectional curvature at $x \in N - \{o\}$ is bounded by the o_w-radial sectional curvatures in M_w^m as follows:

$$\mathcal{K}(\sigma(x)) = K_{o,N}(\sigma_x) \leq -\frac{w''(r)}{w(r)} . \tag{2.5}$$

Then we have for every smooth function $f(r)$ with $f'(r) \leq 0$ for all r, (respectively $f'(r) \geq 0$ for all r):

$$\begin{aligned} \Delta^P(f\circ r) \leq (\geq) \; & (f''(r) - f'(r)\eta_w(r))\,\|\nabla^P r\|^2 \\ & + mf'(r)\left(\eta_w(r) + \langle \nabla^N r, H_P \rangle\right) , \end{aligned} \tag{2.6}$$

where H_P denotes the mean curvature vector of P in N.

2.4. **The Isoperimetric Comparison space.** We are going to define a new kind of model spaces, M_W^m. The limit $\lim_{r\to\infty}\dfrac{W'(r)}{W(r)}$ of the quotient determined by its warping function (this quotient is given in terms of the mean curvature of the geodesic spheres in M_W^m and the bounds on the mean curvature of the submanifold P) will serve as estimate for the isoperimetric constant $\mathcal{I}_\infty(P)$.

Definition 2.6 ([17]). Given the smooth functions $w : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and $h : \mathbb{R}_+ \longrightarrow \mathbb{R}$ with $w(0) = 0$, $w'(0) = 1$ and $-\infty < h(0) < \infty$, the *isoperimetric comparison space* M_W^m is the $W-$model space with base interval $B = [0, R]$ and warping function $W(r)$ defined by the following differential equation:

$$\frac{W'(r)}{W(r)} = \eta_w(r) - \frac{m}{m-1}h(r) . \tag{2.7}$$

and the following boundary condition:

$$W'(0) = 1 . \tag{2.8}$$

By using equation (2.8), it is straightforward to see that $W(r) = 0$ only at $r = 0$, so M^m_W has a well-defined pole o_W at $r = 0$. Moreover, $W(r) > 0$ for all $r > 0$.

Note that when $h(r) = 0$ for all r, then $W(r) = w(r)$ for all r, so M^m_W becomes a model space with warping function w, M^m_w.

Definition 2.7. The model space M^m_W is $w-$*balanced from above* (with respect to the intermediary model space M^m_w) iff the following holds for all $r \in [0, R]$:

$$\begin{aligned} &\eta_w(r) \geq 0 \\ &\eta'_W(r) \leq 0 \ \forall r \quad . \end{aligned} \tag{2.9}$$

Note that $\eta'_W(r) \leq 0 \ \forall r$ is equivalent to the condition

$$-(m-1)(\eta^2_w(r) + K_w(r)) \leq mh'(r) \quad . \tag{2.10}$$

Definition 2.8. The model space M^m_W is $w-$*balanced from below* (with respect to the intermediary model space M^m_w) iff the following holds for all $r \in [0, R]$:

$$q_W(r)\,(\eta_w(r) - h(r)) \geq 1/m \quad . \tag{2.11}$$

Examples . *The following is a list of examples of isoperimetric comparison spaces and balance.*

(1) *Given the functions $w_b(r)$ and $h(r) = C \geq \sqrt{-b}$, $\forall r > 0$, let us consider $\mathbb{K}^m(b) = M^m_{w_b}$ as an intermediary model space with constant sectional curvature $b < 0$. Then, it is straightforward to check that the model space M^m_W defined from w_b and h as in Definition 2.6 is w_b-balanced from above, and is not w_b-balanced from below.*

(2) *Let M^m_w be a model space, with $w(r) = e^{r^2} + r - 1$. Let us now consider $h(r) = 0 \ \forall r > 0$. In this case, as $h(r) = 0$, then $W(r) = w(r)$, so the isoperimetric comparison space M^m_W agrees with its corresponding intermediary model space M^m_w. Moreover, (see* [16]*),*

$$q_w(r)\eta_w(r) \geq \frac{1}{m} \quad .$$

so M^m_w is w-balanced from below.

However, it is easy to see that $\eta_w(r) = \frac{2re^{r^2}+1}{e^{r^2}+r-1}$ is an increasing function from a given value $r_0 > 0$ and, hence, does not satisfy second inequality in (2.9) and is therefore not w-balanced from above.

(3) *Let $\mathbb{K}^m(b) = M^m_{w_b}$, ($b \leq 0$), be the Euclidean or Hyperbolic space, with warping function $w_b(r)$. Let us consider $h(r) = 0 \ \forall r$. In this context, these spaces are isoperimetric spaces with themselves as intermediary spaces, and satisfy both balance conditions given in definitions 2.7 and 2.8 (see* [16]*).*

2.5. **Comparison Constellations.** We now present the precise settings where our main results take place, and introduce the notion of *comparison constellations*.

Definition 2.9. Let N^n denote a Riemannian manifold with a pole o and distance function $r = r(x) = \mathrm{dist}_N(o, x)$. Let P^m denote a complete and properly immersed submanifold in N^n. Suppose the following conditions are satisfied for all $x \in P^m$ with $r(x) \in [0, R]$:

(a) The o-radial sectional curvatures of N are bounded from above by the o_w-radial sectional curvatures of the $w-$model space M^m_w:

$$\mathcal{K}(\sigma_x) \leq -\frac{w''(r(x))}{w(r(x))} \quad .$$

(b) The o-radial mean curvature of P is bounded from above by a smooth radial function, (the *bounding function*) $h : \mathbb{R}_+ \longrightarrow \mathbb{R}$, ($h(0) \in]-\infty, \infty[$):

$$\mathcal{C}(x) \leq h(r(x)) \quad .$$

Let M^m_W denote the W-model with the specific warping function $W : \pi(M^m_W) \to \mathbb{R}_+$ constructed in Definition 2.6 via w, and h. Then the triple $\{N^n, P^m, M^m_W\}$ is called an *isoperimetric comparison constellation* on the interval $[\,0, R]$.

Examples . *Minimal and non-minimal settings will now be described.*

(1) Minimal submanifolds immersed in an ambient Cartan-Hadamard manifold: let P be a minimal submanifold of a Cartan-Hadamard manifold N, with sectional curvatures bounded above by $b \leq 0$. Let us consider the function $h(r) = 0\ \forall r \geq 0$ as the bounding function for the o-radial mean curvature of P and the functions $w_b(r)$ with $b \leq 0$ as the warping function $w(r)$.

It is straigthforward to see that, under these restrictions, $W = w_b$ and, hence, $M^m_W = \mathbb{K}^m(b)$, so $\{N^n,\ P^m,\ \mathbb{K}^m(b)\}$ is an isoperimetric comparison constellation *on the interval $[\,0, R]$, for all $R > 0$. Here the model space $M^m_W = M^m_{w_b} = \mathbb{K}^m(b)$ is w_b-balanced from above and from below.*

(2) Non-minimal submanifolds immersed in an ambient Cartan-Hadamard manifold. Let us consider again a Cartan-Hadamard manifold N, with sectional curvatures bounded above by $a \leq 0$. Let P^m be a properly immersed submanifold in N such that

$$\mathcal{C}(x) \leq h_{a,b}(r(x)) \quad .$$

where, by fixing $a < b < 0$, we define $h_{a,b}(r) = \frac{m-1}{m}(\eta_{w_a}(r) - \eta_{w_b}(r))\ \forall r > 0$.

Then, it is straightforward to check that $W = w_b$ and, hence, $M^m_W = \mathbb{K}^m(b)$, so $\{N^n, P^m, M^m_W\}$ is an isoperimetric comparison constellation *on the interval $[\,0, R]$, for all $R > 0$. Moreover the model space $M^m_W = M^m_{w_b} = \mathbb{K}^m(b)$ is w_a-balanced from above and from below.*

3. Main results

Before stating our main theorems, we find the upper bounds for the isoperimetric quotient defined as the volume of the extrinsic sphere divided by the volume of the extrinsic ball, in the setting given by the comparison constellations.

Theorem 3.1. *(see* [13], [19], [15]*) Consider an isoperimetric comparison constellation $\{N^n, P^m, M^m_W\}$. Assume that the isoperimetric comparison space M^m_W is w-balanced from below. Then*

$$\frac{\operatorname{Vol}(\partial D_t)}{\operatorname{Vol}(D_t)} \geq \frac{\operatorname{Vol}(S^W_t)}{\operatorname{Vol}(B^W_t)} \quad . \tag{3.1}$$

Furthermore, the function $f(t) = \frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B^W_t)}$ is monotone non-decreasing in t.

Moreover, if equality holds in (3.1) for some fixed radius $t_0 > 0$, then D_{t_0} is a cone in the ambient space N^n.

The following is the upper bound for the Cheeger constant of a submanifold P:

Theorem 3.2. *Consider an isoperimetric comparison constellation $\{N^n, P^m, M^m_W\}$. Assume that the isoperimetric comparison space M^m_W is w-balanced from below. Assume, moreover, that*

(1) $\operatorname{Sup}_{t>0}(\frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B^W_t)}) < \infty$.

(2) The limit $\lim_{t\to\infty} \frac{\operatorname{Vol}(S^W_t)}{\operatorname{Vol}(B^W_t)}$ *exists*

Then

$$\mathcal{I}_\infty(P) \leq \lim_{r\to\infty} \frac{\mathrm{Vol}(S^W_t)}{\mathrm{Vol}(B^W_t)} \quad . \tag{3.2}$$

In particular, let P^m be a complete and minimal submanifold properly immersed in a Cartan-Hadamard manifold N with sectional curvatures bounded from above as $K_N \leq b \leq 0$, and suppose that $\mathrm{Sup}_{t>0}(\frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B^{m,b}_t)}) < \infty$.

Then

$$\mathcal{I}_\infty(P) \leq (m-1)\sqrt{-b} \quad . \tag{3.3}$$

Proof. Let us define

$$F(t) := \frac{\mathrm{Vol}(D_t)'}{\mathrm{Vol}(D_t)} - \frac{\mathrm{Vol}(S^W_t)}{\mathrm{Vol}(B^W_t)} = \left[\ln\left(\frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B^W_t)}\right)\right]' \tag{3.4}$$

By the co-area formula and applying Theorem 3.1 it is easy to see that $F(t)$ is a non-negative function. Moreover, $\frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B^W_t)}$ is non-decreasing (see [15]).

Integrating between $t_0 > 0$ and $t > t_0$:

$$\frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B^W_t)} = \frac{\mathrm{Vol}(D_{t_0})}{\mathrm{Vol}(B^W_{t_0})}\, e^{\int_{t_0}^t F(s)\,ds}$$

But on the other hand, from hypothesis (2) and the fact that $\frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B^W_t)}$ is non-decreasing, we know that $\lim_{t\to\infty} \frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B^W_t)} = \sup_t \frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B^W_t)} < \infty$. Then, since $F(t)) \geq 0\ \forall t > 0$:

$$\int_{t_0}^\infty F(s)ds < \infty$$

and hence there is a monotone increasing sequence $\{t_i\}_{i=0}^\infty$ tending to infinity, such that:

$$\lim_{i\to\infty} F(t_i) = 0 \tag{3.5}$$

Let us consider now the exhaustion $\{D_{t_i}\}_{i=1}^\infty$ of P by these extrinsic balls.

By using equation (1.1), we have that,

$$\mathcal{I}_\infty(P) \leq \frac{\mathrm{Vol}(\partial D_{t_i})}{\mathrm{Vol}(D_{t_i})} \leq \frac{(\mathrm{Vol}(D_{t_i}))'}{\mathrm{Vol}(D_{t_i})} \quad \forall r_i \tag{3.6}$$

On the other hand, since $\lim_{i\to\infty} F(t_i) = 0$, then

$$\lim_{i\to\infty} \frac{(\mathrm{Vol}(D_{t_i}))'}{\mathrm{Vol}(D_{t_i}))} = \lim_{i\to\infty} \frac{\mathrm{Vol}(S^W_{t_i})}{\mathrm{Vol}(B^W_{t_i})} \tag{3.7}$$

and therefore

$$\mathcal{I}_\infty(P) \leq \lim_{i\to\infty} \frac{\mathrm{Vol}(S^W_{t_i})}{\mathrm{Vol}(B^W_{t_i})} \tag{3.8}$$

Inequality (3.3) follows inmediately taking into account that, as was shown in the examples above, when P is minimal in a Cartan-Hadamard manifold, then considering $h(r) = 0\ \ \forall r$ and considering $w(r) = w_b(r)$, we have that $\{N^n, P^m, \mathbb{K}^m(b)\}$ is a comparison constellation, with $\mathbb{K}^m(b)$ w_b-balanced from below.

As by hypothesis, $\operatorname{Sup}_{t>0}(\frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{b,m})}) < \infty$ and we have that,

$$
\begin{aligned}
&\lim_{t\to\infty}\frac{\operatorname{Vol}(S_t^W)}{\operatorname{Vol}(B_t^W)} = \lim_{t\to\infty}\frac{\operatorname{Vol}(S_t^{0,m-1})}{\operatorname{Vol}(B_t^{0,m})} = 0 \quad \text{if } b=0\\
&\lim_{t\to\infty}\frac{\operatorname{Vol}(S_t^W)}{\operatorname{Vol}(B_t^W)} = \lim_{t\to\infty}\frac{\operatorname{Vol}(S_t^{b,m-1})}{\operatorname{Vol}(B_t^{b,m})} = (m-1)\sqrt{-b} \quad \text{if } b<0
\end{aligned}
\tag{3.9}
$$

we now apply inequality (3.2). □

Now, we have the following result, which is a direct extension to Yau's classical result (see [21]) on minimal submanifolds, using the same techniques as in [4]:

Theorem 3.3. *Consider an isoperimetric comparison constellation $\{N^n, P^m, M_W^m\}$. Assume that the isoperimetric comparison space M_W^m is w-balanced from above. Assume, moreover, that the limit $\lim_{r\to\infty}\frac{W'(r)}{W(r)}$ exists.*

Then

$$
\mathcal{I}_\infty(P) \geq (m-1)\lim_{r\to\infty}\frac{W'(r)}{W(r)} \quad . \tag{3.10}
$$

In particular, let P^m be a complete and minimal submanifold properly immersed in a Cartan-Hadamard manifold N with sectional curvatures bounded from above as $K_N \leq b \leq 0$.

Then

$$
\mathcal{I}_\infty(P) \geq (m-1)\sqrt{-b} \quad . \tag{3.11}
$$

Proof. From equation (2.7) in definition 2.6 of the isoperimetric comparison space, we have:

$$
(m-1)\frac{W'(r)}{W(r)} + \eta_w(r) = m\,(\eta_w(r) - h(r)) \tag{3.12}
$$

On the other hand, from Theorem 2.5:

$$
\begin{aligned}
\Delta^P r \;&\geq\; \left(m - \|\nabla^P r\|^2\right)\eta_w(r) + m\langle\, \nabla^N r,\, H_P\,\rangle \geq\\
&(m-1)\eta_w(r) + m\langle\, \nabla^N r,\, H_P\,\rangle \geq\\
&(m-1)\eta_w(r) - m\,h(r) =\\
&m\,(\eta_w(r) - h(r)) - \eta_w(r)
\end{aligned}
\tag{3.13}
$$

Then, applying (3.12)

$$
\triangle^P r \geq (m-1)\frac{W'(r)}{W(r)} \tag{3.14}
$$

Now, if we consider a domain $\Omega \subseteq P$, which is precompact and with smooth closure, we have, given its outward unitary normal vector field, ν:

$$
\langle \nu, \nabla^P r\rangle \leq 1
$$

hence by applying divergence Theorem, and taking into account that $\frac{W'(r)}{W(r)}$ is non-increasing

$$
\begin{aligned}
&\operatorname{Vol}(\partial\Omega) \geq \int_{\partial\Omega}\langle \nu, \nabla^P r\rangle d\mu\\
&= \int_\Omega \Delta^P r d\sigma \geq \int_\Omega \frac{W'(r)}{W(r)}d\sigma \geq (m-1)\lim_{r\to\infty}\frac{W'(r)}{W(r)}\operatorname{Vol}(\Omega)
\end{aligned}
\tag{3.15}
$$

As

$$
\frac{\operatorname{Vol}(\partial\Omega)}{\operatorname{Vol}(\Omega)} \geq (m-1)\lim_{r\to\infty}\frac{W'(r)}{W(r)}
$$

for any domain Ω, we have the result.

Inequality (3.11) follows inmediately taking into account that, as in the proof of Theorem 3.2 and in the examples above, when P is minimal in a Cartan-Hadamard manifold, then we have that $\{N^n, P^m, \mathbb{K}^m(b)\}$ is a comparison constellation ($h(r) = 0\ \ \forall r$ and $w(r) = w_b(r)$), with the isoperimetric comparison space used as a model $M^m_W = \mathbb{K}^m(b)$ w_b-balanced from above. Moreover, $\lim_{r\to\infty} \frac{W'(r)}{W(r)} = \sqrt{-b}$. □

4. Applications: Cheeger constant of minimal submanifolds of Cartan-Hadamard manifolds

4.1. **Isoperimetric results and Chern-Osserman Inequality.** This subsection provides two results which describe how minimality and the control on the total extrinsic curvature of the submanifold implies, among other topological consequences, having finite volume growth. The first (Theorem 4.1) is due to M.T. Anderson, and the second (Theorem 4.2) was proved in the Euclidean setting by S.S. Chern and R. Osserman, with an extension to the Hyperbolic setting due to Q. Chen. These results will be used to prove Corollaries 4.4 and 4.5 in the next Subsection §4.2.

Theorem 4.1. *(see* [1]*). Let P^m be an oriented, connected and complete minimal submanifold immersed in the Euclidean space $\mathbb{R}^n$. Let us suppose that $\int_P \|B^P\|^m d\sigma < \infty$, where B^P is the second fundamental form of P. Then*

(1) P has finite topological type.
(2) $\operatorname{Sup}_{t>0}(\frac{\operatorname{Vol}(\partial D_t)}{\operatorname{Vol}(S_t^{0,m-1})}) < \infty$.
(3) $-\chi(P) = \int_P \Phi d\sigma + \lim_{t\to\infty} \frac{\operatorname{Vol}(\partial D_t)}{\operatorname{Vol}(S_t^{0,m-1})}$.

where $\chi(P)$ is the Euler characteristic of P and Φ is the Gauss-Bonnet-Chern form on P, and $S_t^{b,m-1}$ denotes the geodesic t-sphere in $\mathbb{K}^m(b)$.

Remark d. Note that, on applying inequality (3.1) in Theorem 3.1 to the submanifold P the theorem above, we conclude that, under the assumptions of Theorem 4.1, we have the following bound for the volume growth

$$\operatorname{Sup}_{t>0}(\frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{0,m})}) \leq \operatorname{Sup}_{t>0}(\frac{\operatorname{Vol}(\partial D_t)}{\operatorname{Vol}(S_t^{0,m-1})}) < \infty \quad . \tag{4.1}$$

where $B_t^{b,m}$ denotes the geodesic t-ball in $\mathbb{K}^m(b)$.

On the other hand, we have that Chern-Osserman Inequality is satisfied by complete and minimal surfaces in a simply connected real space form with constant sectional curvature $b \leq 0$, $\mathbb{K}^n(b)$. Namely

Theorem 4.2. *(see* [1], [6] *and* [8]*. For an alternative proof, see* [9]*). Let P^2 be an complete minimal surface immersed in a simply connected real space form with constant sectional curvature $b \leq 0$, $\mathbb{K}^n(b)$. Let us suppose that $\int_P \|B^P\|^2 d\sigma < \infty$. Then*

(1) P has finite topological type.
(2) $\operatorname{Sup}_{t>0}(\frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{b,2})}) < \infty$.
(3) $-\chi(P) \leq \frac{\int_P \|B^P\|^2}{4\pi} - \operatorname{Sup}_{t>0} \frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B_t^{b,2})}$.

where $\chi(P)$ is the Euler characteristic of P.

4.2. **The Corollaries.** In this subsection, we are going to state and prove the following results, which are direct consequences of the main theorems in Section §.3 and Theorems 4.1 and 4.2 in Subsection §4.1.

The first Corollary 4.3 is a direct application of Theorems 3.2 and 3.3.

Corollary 4.3. *Let P^m be a complete and minimal submanifold properly immersed in a Cartan-Hadamard manifold N with sectional curvatures bounded from above as $K_N \leq b \leq 0$. Let us suppose that $\mathrm{Sup}_{t>0}(\frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B_t^{b,m})}) < \infty$*

Then

$$\mathcal{I}_\infty(P) = (m-1)\sqrt{-b} \quad . \tag{4.2}$$

Proof. This is a direct consequence of inequalities (3.3) and (3.11) in Theorem 3.2 and Theorem 3.3. □

The second and the third corollaries 4.4 and 4.5 are based on Theorems 4.1 and 4.2.

When we consider minimal submanifolds in $\mathbb{R}^n$, we have the following result:

Corollary 4.4. *Let P^m be a complete and minimal submanifold properly immersed in $\mathbb{R}^n$, with finite total extrinsic curvature $\int_P \|B^P\|^m d\sigma < \infty$.*

Then

$$\mathcal{I}_\infty(P) = 0 \quad . \tag{4.3}$$

Proof. In this case, taking $h(r) = 0\ \forall r$ and $w_0(r) = r$, we have that $\{\mathbb{R}^n, P^m, \mathbb{R}^m\}$ is a comparison constellation bounded from above, with $\mathbb{R}^m$ w_0-balanced from below. Hence, we apply Theorem 3.1 to obtain

$$\frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B_t^{0,m})} \leq \frac{\mathrm{Vol}(\partial D_t)}{\mathrm{Vol}(S_t^{0,m-1})} \quad \text{for all } t > 0 \quad . \tag{4.4}$$

Therefore, as the total extrinsic curvature of P is finite, by applying Theorem 4.1, inequality (4.4) and Remark d, we have

$$\mathrm{Sup}_{t>0}(\frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B_t^{0,m})}) < \infty$$

Finally,

$$\lim_{t\to\infty} \frac{\mathrm{Vol}(S_t^{0,m-1})}{\mathrm{Vol}(B_t^{0,m})} = \lim_{t\to\infty} \frac{m}{t} = 0$$

Hence, applying Theorem 3.2, $\mathcal{I}_\infty(P) \leq 0$, so $\mathcal{I}_\infty(P) = 0$. □

Corollary 4.4 can be extended to complete and minimal surfaces (properly) immersed in the Hyperbolic space, with finite total extrinsic curvature:

Corollary 4.5. *Let P^2 be a complete and minimal surface immersed in $\mathbb{K}^n(b)$ with finite total extrinsic curvature $\int_P \|B^P\|^2 d\sigma < \infty$.*

Then

$$\mathcal{I}_\infty(P) = \sqrt{-b} \quad . \tag{4.5}$$

Proof. As the total extrinsic curvature of P is finite, by applying Theorem 4.2 we have:

$$\mathrm{Sup}_{t>0}(\frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B_t^{b,2})}) < \infty$$

Then, apply Corollary 4.3 with $m = 2$. □

REFERENCES

[1] M. T. Anderson, *The compactification of a minimal submanifold in Euclidean space by the Gauss map* , I.H.E.S. Preprint, 1984

[2] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press (1984).

[3] I. Chavel, *Isoperimetric inequalities. Differential geometric and analytic perspectives*, Cambridge Tracts in Mathematics, **145**. Cambridge University Press (2001).

[4] I. Chavel, *Riemannian geometry: A modern introduction*, Cambridge Tracts in Mathematics, **108**. Cambridge University Press (1993).

[5] J. Cheeger, *A Lower Bound for the Smallest Eigenvalue of the Laplacian*, Problems in analysis, (Papers dedicated to Salomon Bochner, 1969), pp 195-199, N.J. 1970.

[6] Q. Chen, *On the volume growth and the topology of complete minimal submanifolds of a Euclidean space* J. Math. Sci. Univ. Tokyo 2 (1995), 657-669.

[7] Q. Chen, *On the area growth of minimal surfaces in* $\mathbb{H}^n$, Geometriae Dedicata, **75** (1999), 263–273.

[8] Q. Chen Qing and Y. Cheng, *Chern-osserman inequality for minimal surfaces in* $\mathbb{H}^n$, Proc. Amer. Math Soc., Vol. 128, **8**, (1999), 2445-2450.

[9] V. Gimeno and V. Palmer, *Extrinsic isoperimetry and compactification of minimal surfaces in Euclidean and Hyperbolic spaces*, Preprint accepted for publication in Israel Jour. Math, (2010), arXiv:1011.5380v1.

[10] R. Greene and S. Wu *Function theory on manifolds which posses a pole*, Lecture Notes in Math., **699**, (1979), Springer Verlag, Berlin.

[11] A. Grigor'yan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*, Bull. Amer. Math. Soc. **36** (1999), 135–249.

[12] D. Hoffman and J. Spruck, *Sobolev and Isoperimetric Inequalities for Riemannian Submanifolds*, Comm. of Pure and App. Mathematics, **27** (1974), 715-727.

[13] A. Hurtado, S. Markvorsen and V. Palmer, *Torsional rigidity of submanifolds with controlled geometry*, Math. Ann. **344** (2009), 511–542.

[14] L.P. Jorge and D. Koutroufiotis, *An estimate for the curvature of bounded submanifolds*, Amer. Journal of Math., **103**, n. 4 (1981), 711-725.

[15] S. Markvorsen and V. Palmer, *The relative volume growth of minimal submanifolds*, Archiv der Mathematik, **79** (2002), 507–514.

[16] S. Markvorsen and V. Palmer, *Torsional rigidity of minimal submanifolds*, Proc. London Math. Soc. **93** (2006), 253–272.

[17] S. Markvorsen and V. Palmer, *Extrinsic isoperimetric analysis on submanifolds with curvatures bounded from below*, J. Geom. Anal. **20** (2010) 388–421.

[18] B. O'Neill, *Semi-Riemannian Geometry; With Applications to Relativity*, Academic Press (1983).

[19] V. Palmer, *Isoperimetric Inequalities for extrinsic balls in minimal submanifolds and their applications*, J. London Math. Soc. (2) **60** (1999), 607-616.

[20] V. Palmer, *On deciding whether a submanifold is parabolic of hyperbolic using its mean curvature* Simon Stevin Transactions on Geometry, vol 1. 131-159, Simon Stevin Institute for Geometry, Tilburg, The netherlands, 2010.

[21] S.T. Yau, *Isoperimetric constants and the first eigenvalue of a compact Riemannian manifold*, Annales Scientifiques de L'E.N.S., **8**, num. 4, (1975), 487-507.

DEPARTAMENT DE MATEMÀTIQUES-INIT, UNIVERSITAT JAUME I, CASTELLÓ, SPAIN.
E-mail address: gimenov@guest.uji.es

DEPARTAMENT DE MATEMÀTIQUES-INIT, UNIVERSITAT JAUME I, CASTELLÓ, SPAIN.
E-mail address: palmer@mat.uji.es

5

Creixement del volum, nombre de finals i la topologia de subvarietats completes

- Article publicat en Arxiv.org com *arXiv:1112.4042v2 [math.DG]*.
- Article sota revisió per a la seua possible publicació.

VOLUME GROWTH, NUMBER OF ENDS AND THE TOPOLOGY OF A COMPLETE SUBMANIFOLD

VICENT GIMENO* AND VICENTE PALMER*

Abstract. Given a complete isometric immersion $\varphi : P^m \longrightarrow N^n$ in an ambient Riemannian manifold N^n with a pole and with radial sectional curvatures bounded from above by the corresponding radial sectional curvatures of a radially symmetric space M^n_w, we determine a set of conditions on the extrinsic curvatures of P that guarantees that the immersion is proper and that P has finite topology in the line of the results in [24] and [25]. When the ambient manifold is a radially symmetric space, it is shown an inequality between the (extrinsic) volume growth of a complete and minimal submanifold and its number of ends which generalizes the classical inequality stated in [1] for complete and minimal submanifolds in $\mathbb{R}^n$. We obtain as a corollary the corresponding inequality between the (extrinsic) volume growth and the number of ends of a complete and minimal submanifold in the Hyperbolic space together with Bernstein type results for such submanifolds in Euclidean and Hyperbolic spaces, in the vein of the work [12].

1. Introduction

A natural question in Riemannian geometry is to explore the influence of the curvature conduct of a complete Riemannan manifold on its geometric and topological properties. Classical results concernig this are the gap theorems showed by Greene and Wu in [7], (see too [8]), and, when it is considered a minimal submanifold (properly) immersed in the Euclidean space $\mathbb{R}^n$, the Berstein-type theorems showed by Anderson in [1] and by Schoen in [32]. Greene and Wu's results states, roughly speaking, that a Riemannian manifold with a pole and with faster than quadratic decay of its sectional curvatures is isometric to the Euclidean space. On the other hand, Anderson proved, as a corollary of a generalization of the Chern-Osserman theorem on complete and minimal submanifolds of $\mathbb{R}^n$ with finite total (extrinsic) curvature, that any of such submanifolds having one end is an affine n-plane. More examples concerning submanifolds immersed in an ambient Riemannian manifold and the analysis of its (intrinsic and extrinsic) curvature behavior are the gap results, (of Bernstein-type), given by Kasue and Sugahara in [12] (see Theorems A and B), where an accurate (extrinsic) curvature decay forces to minimal, (or not) submanifolds with one end of the Euclidean and Hyperbolic spaces to be totally

2000 *Mathematics Subject Classification.* Primary 53A20 53C40; Secondary 53C42.

Key words and phrases. volume growth, minimal submanifold, end, Hessian-Index comparison theory, extrinsic distance, total extrinsic curvature, second fundamental form, gap theorem, Bernstein-type theorem.

* Work partially supported by the Caixa Castelló Foundation, and DGI grant MTM2010-21206-C02-02.

geodesic, and the gap results for minimal submanifolds in the Euclidean space with controlled scalar curvature given by Kasue in [13].

The estimation of the number of ends of these submanifolds plays a fundamental rôle in all the Bernstein-type results above mentioned. In this way, it is proved in [1] (see Theorems 4.1 and 5.1 in that paper) that given a complete and minimal submanifold $\varphi : P^m \longrightarrow \mathbb{R}^n$, $(m > 2)$ having finite total curvature $\int_P \|B^P\|^m d\sigma < \infty$, its (extrinsic) volume growth, defined as the quotient $\frac{\operatorname{Vol}(\varphi(P)\cap B_t^{0,n})}{\omega_n t^n}$ is bounded from above by the number of ends of P, $\mathcal{E}(P)$, namely

$$\lim_{t\to\infty} \frac{\operatorname{Vol}(\varphi(P)\cap B_t^{0,n})}{\omega_n t^n} \leq \mathcal{E}(P) \tag{1.1}$$

where $B_t^{b,n}$ denotes the metric $t-$ ball in the real space form of constant curvature b, $\mathbb{K}^n(b)$, and $\|B^P\|$ denotes the Hilbert-Schmidt norm of the second fundamental form of P in $\mathbb{R}^n$. If moreover $\mathcal{E}(P) = 1$, it is concluded (using inequality (1.1)) the Bernstein-type result above alluded, namely, that P^m is an affine plane, i.e. totally geodesic in $\mathbb{R}^n$, (see Theorem 5.2 in [1]).

In the paper [3] it was proved that inequality (1.1) is in fact an equality when the minimal submanifold in $\mathbb{R}^n$ exhibits an accurate decay of its extrinsic curvature $\|B^P\|$ and in the paper [12] it was proved that, if the submanifold P has only one end and the decay of its extrinsic curvature $\|B^P\|$ is faster than linear, (when the ambient space is $\mathbb{R}^n$) or than exponential, (when the ambient space is $\mathbb{H}^n(b)$), then it is is totally geodesic.

Within this study of the behavior at infinity of complete and minimal submanifolds with finite total curvature immersed in the Euclidean space, it was proved also in [1] and in [22] that the immersion of a complete and minimal submanifold P in $\mathbb{R}^n$ or $\mathbb{H}^n(b)$ satisfying $\int_P \|B^P\|^m d\sigma < \infty$ is proper and that P is of finite topological type.

We should mention here the results in [24] and in [25], where has been stated new conditions on the decay of the extrinsic curvature for a completely immersed submanifold P in the Euclidean space ([24]) and in a Cartan-Hadamard manifold ([25]) which guarantees the properness of the submanifold and the finiteness of its topology.

In view of these results, it seems natural to consider the following three issues:

(1) Can the properness/finiteness results in [24] and [25] be extended to submanifolds immersed in spaces which have not necessarily non-positive curvature?,
(2) Do we have an analogous to inequality (1.1) between the extrinsic volume growth and the number of ends when we consider a minimal submanifold (properly) immersed in Hyperbolic space which exhibit an accurate extrinsic curvature decay?.
(3) Moreover, is it possible to deduce from this inequality a Bernstein-type result in the line of [1] and [12]?.

We provide in this paper a (partial) answer to these questions, besides other lower bounds for the number of ends for (non-minimal) submanifolds in the Euclidean and Hyperbolic spaces and other gap results related with these estimates. As a preliminary view of our results, we have the following theorems, Theorem 1.1 and Theorem 1.2, which follows directly from our Theorem 3.5. In Theorem 1.1 we have the answer to the two last questions, namely, setting equation (1.1), but in the Hyperbolic case, and a Bernstein-type result for minimal submanifolds in the Hyperbolic space, in the line studied by Kasue and Sugahara in [12], (see assertion (A-iv) of Theorem A). On the other hand, Theorem 1.2 encompasses a slightly less general version of assertion (A-i) of Theorem A in [12].

Theorem 1.1. *Let $\varphi : P^m \longrightarrow \mathbb{H}^n(b)$ be a complete, proper and minimal immersion with $m > 2$. Let us suppose that for sufficiently large R_0 and for all points $x \in P$ such that $r(x) > R_0$, (i.e. outside a compact),*

$$\|B^P_x\| \leq \frac{\delta(r(x))}{e^{2\sqrt{-b}\, r(x)}}$$

where $r(x) = d_{\mathbb{H}^n(b)}(o, \varphi(x))$ is the (extrinsic) distance in $\mathbb{H}^n(b)$ of the points in $\varphi(P)$ to a fixed pole $o \in \mathbb{H}^n(b)$ such that $\varphi^{-1}(o) \neq \emptyset$ and $\delta(r)$ is a smooth function such that $\delta(r) \to 0$ when $r \to \infty$. Then:

(1) The finite number of ends $\mathcal{E}(P)$ is related with the volume growth by

$$\mathrm{Sup}_{t>0} \frac{D_t(o)}{\mathrm{Vol}(B^{b,m}_t)} \leq \mathcal{E}(P)$$

where $D_t(o) = \{x \in P : r(x) < t\} = \{x \in P : \varphi(x) \in B^{b,n}_t(o)\}$ is the extrinsic ball of radius t in P, (see Definition 2.1).

(2) If P has only one end, P is totally geodesic in $\mathbb{H}^n(b)$

When the ambient manifold is $\mathbb{R}^n$, we have the following Bernstein-type result as in [12]:

Theorem 1.2. *Let $\varphi : P^m \longrightarrow \mathbb{R}^n$ be a complete non-compact, minimal and proper immersion with $m > 2$. Let us suppose that for sufficiently large R_0 and for all points $x \in P$ such that $r(x) > R_0$, (i.e. outside the compact extrinsic ball $D_{R_0}(o)$ with $\varphi^{-1}(o) \neq \emptyset$),*

$$\|B^P_x\| \leq \frac{\epsilon(r(x))}{r(x)}$$

where $\epsilon(r)$ is a smooth function such that $\epsilon(r) \to 0$ when $r \to \infty$. Then:

(1) The finite number of ends $\mathcal{E}(P)$ is related with the volume growth by

$$\mathrm{Sup}_{t>0} \frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B^{0,m}_t)} \leq \mathcal{E}(P)$$

(2) If P has only one end, P is totally geodesic in $\mathbb{R}^n$.

These results, that we shall prove in Section 8, (together the corollaries of Section 4), follows from two main theorems, stablished in Section 3. In the first (Theorem 3.1) we show that a complete isometric immersion $\varphi : P^m \longrightarrow N^n$, $(m > 2)$, with controlled second fundamental form in a complete Riemannian manifold which possess a pole and has controlled radial sectional curvatures is proper and has finite topology. In the second (Theorem 3.4) it is proved that a complete and proper isometric immersion $\varphi : P^m \longrightarrow M^n_w$, $(m > 2)$, with controlled second fundamental form in a radially symmetric space M^n_w with sectional curvatures bouded from below by a radial function has its volume growth bounded from above by a quantity which involve its (finite) number of ends.

The proof of both theorems follows basically the argumental lines of the proofs given in [24] and [25] and some ideas in [3]. An important difference to these results is that, on our side, we allow to the ambient manifold to have positive sectional curvatures, bounding from above only the sectional curvatures of the planes containing radial directions. However, to show the properness of the immersion in [25], the ambient manifold must have non-positive sectional curvatures, and to assure the finiteness of the topology of the immersion P, this ambient manifold must be, in addition, simply connected, (i.e. a Cartan-Hadamard manifold). This difference is based in following considerations.

To obtain the finiteness of the topology in Theorem 3.1, we show that the restricted, (to the submanifold) extrinsic distance to a fixed pole (in the ambient manifold) has no critical points outside a compact and then, we apply classical Morse theory. To show that the extrinsic distance function has no critical points we compute its Hessian as we can find it in [16] and [27]. These results are, in its turn, based in the Jacobi-Index analysis for the Hessian of the distance function given in [6], in particular, its Theorem A, (see Subsection 2.3). This comparison theorem is different of the Hessian comparison Theorem 1.2 used in [25]: while in this last theorem, the space used as a model to compare is the real space form with constant sectional curvature equal to the bound on the sectional curvatures of the given Riemannian manifold, in our adaptation of Theorem A in [6], (see Theorem 2.10), only the sectional curvatures of the planes containing radial directions from the pole are bounded by the corresponding radial sectional curvatures in a radially symmetric space used as a model.

We also note at this point that although we use the definition of pole given by Greene and Wu in [6], (namely, the exponential must be a diffeomorphism at a pole), in fact, the comparison of the Hessians in Theorem A holds along radial geodesics from the poles defined as those points which have not conjugate points, as in [25].

1.1. **Outline.** The outline of the paper is the following. In Section §.2 we present the definiton of extrinsic ball, together the basic facts about the Hessian comparison theory of restricted distance function we are going to use and an isoperimetric inequality for the extrinsic balls which plays an important rôle in the proof of Theorem 3.4 . Section §.3 is devoted to the statement of the main results (Theorem 3.1, Theorem 3.4 and Theorem 3.5). We shall present in Section 4 two lists of results based in Theorems 3.1, 3.4 and 3.5: the first set of consequences is devoted to bound from above the volume growth of a submanifold by the number of its ends, in several contexts, obtaining moreover some Bernstein-type results. In the second set of corollaries are stated some compactification theorems for submanifolds in $\mathbb{R}^n$, in $\mathbb{H}^n$ and in $\mathbb{H}^n \times \mathbb{R}^l$. Sections §.5, §.6, §.7 are devoted to the proof of Theorems 3.1, 3.4, and 3.5, respectively. Theorem 1.1, Theorem 1.2 and the corollaries stated in Section §.4 are proved in Section §.8.

2. Preliminaires

2.1. **The extrinsic distance.** We assume throughout the paper that $\varphi : P^m \longrightarrow N^n$ is an isometric immersion of a complete non-compact Riemannian m-manifold P^m into a complete Riemannian manifold N^n with a pole $o \in N$, (this is the precise meaning we shall give to the word *submanifold* along the text) . Recall that a pole is a point o such that the exponential map

$$\exp_o\colon T_oN^n \to N^n$$

is a diffeomorphism. For every $x \in N^n - \{o\}$ we define $r(x) = r_o(x) = \text{dist}_N(o, x)$, and this distance is realized by the length of a unique geodesic from o to x, which is the *radial geodesic from* o. We also denote by $r|_P$ or by r the composition $r \circ \varphi : P \to \mathbb{R}_+ \cup \{0\}$. This composition is called the *extrinsic distance function* from o in P^m. The gradients of r in N and $r|_P$ in P are denoted by $\nabla^N r$ and $\nabla^P r$, respectively. Then we have the following basic relation, by virtue of the identification, given any point $x \in P$, between the tangent vector fields $X \in T_xP$ and $\varphi_{*_x}(X) \in T_{\varphi(x)}N$

$$\nabla^N r = \nabla^P r + (\nabla^N r)^\perp, \tag{2.1}$$

where $(\nabla^N r)^\perp(\varphi(x)) = \nabla^\perp r(\varphi(x))$ is perpendicular to T_xP for all $x \in P$.

Definition 2.1. Given $\varphi : P^m \longrightarrow N^n$ an isometric immersion of a complete and connected Riemannian m-manifold P^m into a complete Riemannian manifold N^n with a pole $o \in N$, we denote the *extrinsic metric balls* of radius $t > 0$ and center $o \in N$ by $D_t(o)$. They are defined as the subset of P:

$$D_t(o) = \{x \in P : r(\varphi(x)) < t\} = \{x \in P : \varphi(x) \in B^N_t(o)\}$$

where $B^N_t(o)$ denotes the open geodesic ball of radius t centered at the pole o in N^n. Note that the set $\varphi^{-1}(o)$ can be the empty set.

Remark 2.2. When the imersion φ is proper, the extrinsic domains $D_t(o)$ are precompact sets, with smooth boundary $\partial D_t(o)$. The assumption on the smoothness of $\partial D_t(o)$ makes no restriction. Indeed, the distance function r is smooth in $N - \{o\}$ since N is assumed to possess a pole $o \in N$. Hence the composition $r|_P$ is smooth in P and consequently the radii t that produce smooth boundaries $\partial D_t(o)$ are dense in $\mathbb{R}$ by Sard's theorem and the Regular Level Set Theorem.

We now present the curvature restrictions which constitute the geometric framework of our study.

Definition 2.3. Let o be a point in a Riemannian manifold N and let $x \in N - \{o\}$. The sectional curvature $K_N(\sigma_x)$ of the two-plane $\sigma_x \in T_xN$ is then called a *o-radial sectional curvature* of N at x if σ_x contains the tangent vector to a minimal geodesic from o to x. We denote these curvatures by $K_{o,N}(\sigma_x)$.

2.2. **Model spaces.** Throughout this paper we shall assume that the ambient manifold N^n has its o-radial sectional curvatures $K_{o,N}(x)$ bounded from above by the expression $K_w(r(x)) = -w''(r(x))/w(r(x))$, which are precisely the radial sectional curvatures of the *w-model space* M^m_w we are going to define.

Definition 2.4 (See [23], [10] and [6]). A $w-$model M^m_w is a smooth warped product with base $B^1 = [0, \Lambda[\subset \mathbb{R}$ (where $0 < \Lambda \leq \infty$), fiber $F^{m-1} = \mathbb{S}^{m-1}_1$ (i.e. the unit $(m-1)$-sphere with standard metric), and warping function $w\colon [0, \Lambda[\to \mathbb{R}_+ \cup \{0\}$, with $w(0) = 0$, $w'(0) = 1$, and $w(r) > 0$ for all $r > 0$. The point $o_w = \pi^{-1}(0)$, where π denotes the projection onto B^1, is called the *center point* of the model space. If $\Lambda = \infty$, then o_w is a pole of M^m_w.

Proposition 2.5. *The simply connected space forms $\mathbb{K}^m(b)$ of constant curvature b are $w-$models with warping functions*

$$w_b(r) = \begin{cases} \frac{1}{\sqrt{b}}\sin(\sqrt{b}\,r) & \text{if } b > 0 \\ \quad r & \text{if } b = 0 \\ \frac{1}{\sqrt{-b}}\sinh(\sqrt{-b}\,r) & \text{if } b < 0. \end{cases}$$

Note that for $b > 0$ the function $Q_b(r)$ admits a smooth extension to $r = \pi/\sqrt{b}$.

Proposition 2.6 (See Proposition 42 in Chapter 7 of [23]. See also [6] and [10]). *Let M_w^m be a $w-$model with warping function $w(r)$ and center o_w. The distance sphere S_r^w of radius r and center o_w in M_w^m is the fiber $\pi^{-1}(r)$. This distance sphere has the constant mean curvature $\eta_w(r) = \frac{w'(r)}{w(r)}$.*

On the other hand, the o_w-radial sectional curvatures of M_w^m at every $x \in \pi^{-1}(r)$ (for $r > 0$) are all identical and determined by

$$K_{o_w,M_w}(\sigma_x) = -\frac{w''(r)}{w(r)}.$$

and the sectional curvatures of M_w^m at every $x \in \pi^{-1}(r)$ (for $r > 0$) of the tangent planes to the fiber S_r^w are also all identical and determined by

$$K(r) = K_{M_w}(\Pi_{S_r^w}) = \frac{1 - (w'(r))^2}{w^2(r)}.$$

Remark 2.7. The $w-$model spaces are completely determined via w by the mean curvatures of the spherical fibers S_r^w:

$$\eta_w(r) = w'(r)/w(r) \quad ,$$

by the volume of the fiber

$$\operatorname{Vol}(S_r^w) = V_0\, w^{m-1}(r) \quad ,$$

and by the volume of the corresponding ball, for which the fiber is the boundary

$$\operatorname{Vol}(B_r^w) = V_0 \int_0^r w^{m-1}(t)\, dt \quad .$$

Here V_0 denotes the volume of the unit sphere $S_1^{0,m-1}$, (we denote in general as $S_r^{b,m-1}$ the sphere of radius r in the real space form $\mathbb{K}^m(b)$) . The latter two functions define the isoperimetric quotient function as follows

$$q_w(r) = \operatorname{Vol}(B_r^w)/\operatorname{Vol}(S_r^w) \quad .$$

Besides the rôle of comparison controllers for the radial sectional curvatures of N^n, we shall need two further purely intrinsic conditions on the model spaces:

Definition 2.8. A given $w-$model space M_w^m is called balanced from below and balanced from above, respectively, if the following weighted isoperimetric conditions are satisfied:

Balance from below: $q_w(r)\,\eta_w(r) \geq 1/m$ for all $r \geq 0$;

Balance from above: $q_w(r)\,\eta_w(r) \leq 1/(m-1)$ for all $r \geq 0$.

A model space is called *totally balanced* if it is balanced both from below and from above.

Remark 2.9. If $K_w(r) \geq -\eta_w^2(r)$ then M_w^m is balanced from above. If $K_w(r) \leq 0$ then M_w^m is balanced from below, see the paper [16] for a detailed list of examples.

2.3. **Hessian comparison analysis.** The 2.nd order analysis of the restricted distance function $r_{|_P}$ defined on manifolds with a pole is governed by the Hessian comparison Theorem A in [6].

This comparison theorem can be stated as follows, when one of the spaces is a model space M_w^m, (see [27]):

Theorem 2.10 (See [6], Theorem A)**.** *Let $N = N^n$ be a manifold with a pole o, let $M = M_w^m$ denote a $w-$model with center o_w. Suppose that every o-radial sectional curvature at $x \in N \setminus \{o\}$ is bounded from above by the o_w-radial sectional curvatures in M_w^m as follows:*

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r)}{w(r)}$$

for every radial two-plane $\sigma_x \in T_xN$ at distance $r = r(x) = \operatorname{dist}_N(o,x)$ from o in N. Then the Hessian of the distance function in N satisfies

$$\begin{aligned} \operatorname{Hess}^N(r(x))(X,X) &\geq \operatorname{Hess}^M(r(y))(Y,Y) \\ &= \eta_w(r)\left(\|X\|^2 - \langle \nabla^M r(y), Y\rangle_M^2\right) \\ &= \eta_w(r)\left(\|X\|^2 - \langle \nabla^N r(x), X\rangle_N^2\right) \end{aligned} \tag{2.2}$$

for every vector X in T_xN and for every vector Y in T_yM with $r(y) = r(x) = r$ and $\langle \nabla^M r(y), Y\rangle_M = \langle \nabla^N r(x), X\rangle_N$.

Remark 2.11. As we mentioned in the Introduction, inequality (2.2) is true along the geodesics emanating from o and o_w which are free of conjugate points of o and o_w, (see Remark 2.3 in [6]). Other relevant observation is that the bound given in inequality (2.2) does not depend on the dimension of the model space, (see Remark 3.7 in [27]).

We present now a technical result concerning the Hessian of a radial function, namely, a function which only depends on the distance function r. For the proof of this result, and the rest of the results in this subsection, we refer to the paper [27].

Proposition 2.12. *Let $N = N^n$ be a manifold with a pole o. Let $r = r(x) = \operatorname{dist}_N(o,x)$ be the distance from o to x in N. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ a smooth function. Then, given $q \in N$ and $X, Y \in T_qN$,*

$$\begin{aligned} \operatorname{Hess}^N F \circ r|_q(X,Y) &= F''(r)(\nabla^N r \otimes \nabla^N r)(X,Y) \\ &\quad + F'(r)\operatorname{Hess}^N r|_q(X,Y) \end{aligned} \tag{2.3}$$

Now, let us consider a complete isometric immersion $\varphi : P^m \longrightarrow N$ in a Riemannian ambient manifold N^n with pole o, and with distance function to the pole r. We are going to see how the Hessians (in P and in N), of a radial function defined in the submanifold are related via the second fundamental form B^P of the submanifold P in N. As before, we identify, given any $q \in P$, the tangent vectors $X \in T_qP$ with $\varphi_{*_q}X \in T\varphi(q)N$ along the next results.

Proposition 2.13. *Let N^n be a manifold with a pole o, and let us consider an isometric immersion $\varphi : P^m \longrightarrow N$. If $r|_P$ is the extrinsic distance function, then, given $q \in P$ and $X, Y \in T_qP$,*

$$\text{Hess}^P r|_q(X,Y) = \text{Hess}^N r|_{\varphi(q)}(X,Y) + \langle B^P_q(X,Y), \nabla^N r|_q \rangle \tag{2.4}$$

where B^P_q is the second fundamental form of P in N at the point $q \in P$.

Now, we apply Proposition 2.12 to $F \circ r|_P = F \circ r \circ \varphi$, (considering P as the Riemannian manifold where the function is defined), to obtain an expression for $\text{Hess}^P F \circ r|_P(X,Y)$. Then, let us apply Proposition above to $\text{Hess}^P r|_P(X,Y)$, and we finally get:

Proposition 2.14. *Let $N = N^n$ be a manifold with a pole o, and let P^m denote an immersed submanifold in N. Let $r|_P$ be the extrinsic distance function. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth function. Then, given $q \in P$ and $X, Y \in T_qP$,*

$$\begin{aligned} \text{Hess}^P F \circ r|_q(X,Y) = {} & F''(r(q)) \langle \nabla^N r|_q, X \rangle \langle \nabla^N r|_q, Y \rangle \\ & + F'(r(q)) \{ \text{Hess}^N r|_q(X,Y) \\ & + \langle \nabla^N r|_q, B^P_q(X,Y) \rangle \} \end{aligned} \tag{2.5}$$

2.4. **Comparison constellations and Isoperimetric inequalities.** The isoperimetric inequalities satisfied by the extrinsic balls in minimal submanifolds are on the basis of the monotonicity of the volume growth function $f(r) = \frac{Vol(D_r)}{Vol(B^w_r)}$, a key result to prove Theorem 1.1. We have the following theorem.

Theorem 2.15 (See [16], [17], [18], [19] and [26]). *Let $\varphi : P^m \longrightarrow N^n$ be a complete, proper and minimal immersion in an ambient Riemannian manifold N^n which possess at least one pole $o \in N$. Let us suppose that the $o-$radial sectional curvatures of N are bounded from above by the o_w-radial sectional curvatures of the $w-$model space M^m_w:*

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r(x))}{w(r(x))} \quad \forall x \in N$$

and assume that M^m_w is balanced from below. Let D_r be an extrinsic r-ball in P^m, with center at a pole $o \in N$ in the ambient space N. Then:

$$\frac{\text{Vol}(\partial D_r)}{\text{Vol}(D_r)} \geq \frac{\text{Vol}(S^w_r)}{\text{Vol}(B^w_r)} \quad \textit{for all } r > 0 \quad . \tag{2.6}$$

Furthermore, if $\varphi^{-1}(o) \neq \emptyset$,

$$\text{Vol}(D_r) \geq \text{Vol}(B^w_r) \quad \textit{for all } r > 0 \quad . \tag{2.7}$$

Moreover, if equality in inequalities (2.6) or (2.7) holds for some fixed radius R and if the balance of M^m_w from below is sharp $q_w(r)\,\eta_w(r) > 1/m$ for all r, then D_R is a minimal cone in the ambient space N^n, so if N^n is the hyperbolic space $\mathbb{H}^n(b)$, $b < 0$, then P^m is totally geodesic in $\mathbb{H}^n(b)$.

If, on the other hand, the ambient space is $\mathbb{R}^n$ and equality in inequalities (2.6) or (2.7) holds for all radius $r > 0$ then P^m is totally geodesic in $\mathbb{R}^n$.

On the other hand, and also as a consequence of inequality (2.6), the volume growth function $f(r) = \frac{Vol(D_r)}{Vol(B^w_r)}$ is a non-decreasing function of r.

3. Main Results

We prove in this section our main results, stablishing a set of conditions that assures that our submanifolds are properly immersed and have finite topology and bounding from below, under certain conditions, the number of its ends.

Theorem 3.1. *Let $\varphi : P^m \longrightarrow N^n$ be an isometric immersion of a complete non-compact Riemannian m-manifold P^m into a complete Riemannian manifold N^n with a pole $o \in N$ and satisfying $\varphi^{-1}(o) \neq \emptyset$. Let us suppose that:*

(1) The $o-$radial sectional curvatures of N are bounded from above by the o_w-radial sectional curvatures of the $w-$model space M^m_w:

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r(x))}{w(r(x))} \ \ \forall x \in N.$$

(2) The second fundamental form B^P_x in $x \in P$ satisfies that, for sufficiently large radius R_0, and for some constant $c \in]0,1[$:

$$\|B^P_x\| \leq c\,\eta_w(\rho^P(x)) \ \ \forall x \in P - B^P_{R_0}(x_o)$$

where $\rho^P(x)$ denotes the intrinsic distance in P from some fixed $x_o \in \varphi^{-1}(o)$ to x.

(3) For any $r > 0$, $w'(r) \geq d > 0$ and $(\eta_w(r))' \leq 0$.

Then P is properly immersed in N and it is C^∞- diffeomorphic to the interior of a compact smooth manifold $\overline{P}$ with boundary.

Remark 3.2. To show that φ is proper, we shall use Theorem 2.10. Hence, it is enough to assume that o is a pole in the sense that there are not conjugate points along any geodesic emanating from o, (see [5] and [30]). Therefore our statement about the properness of the immersion includes ambient manifolds N that admit non-negative sectional curvatures, unlike the ambient manifold in Theorem 1.2 in [25]. On the other hand, to prove the finiteness of the topology of P we need to assume that the ambient manifold N posses a pole as it is defined in [6], namely, a point $p \in N$ where exp_p is a C^∞ diffeomorphism. However, although our ambient manifold must be diffeomorphic to $\mathbb{R}^n$ in this case, (as in Theorem 1.2 in [25], where the ambient space must be a Cartan-Hadamard manifold), also admits non-negative sectional curvatures.

To complete the benchmarking with the hypotheses in [24] and [25], we are going to compare the assumptions (2) and (3) in Theorem 3.1 with the notion of "submanifold with *tamed* second fundamental form" introduced in [24]. It is straightforward to check that if $\varphi : P^m \longrightarrow N^n$ is an immersion of a complete Riemannian m-manifold P^m into a complete Riemannian manifold N^n with sectional curvatures $K_N \leq b \leq 0$, and P has tamed second fundamental form, in the sense of Definition 1.1 in [25], then there exists $R_0 > 0$ such that for all $r \geq R_0$, the quantity

$$a_r := \mathrm{Sup}\{\frac{w_b}{w_b'}(\rho^P(x))\|B^P_x\| : x \in P - B^P_r\}$$

satisfies $a_r < 1$.

Hence, taking $r = R_0$, we have that for all $x \in P - B^P_{R_0}$, and some $c \in (0,1)$,

$$\|B^P_x\| \leq c\eta_{w_b}(\rho^P(x))\,.$$

On the other hand, when $b \leq 0$, then $w_b'(r) \geq 1 > 0\ \forall r > 0$ and $(\eta_{w_b}(r))' \leq 0\ \forall r > 0$.

All these observations make us consider our Theorem 3.1 as a natural and slight generalization of assertions (b) and (c) of Theorem 1.2 in [25].

Observe that if we assume the properness of the immersion we obtain the following version of Theorem 3.1, where we can remove the hypothesis about the decrease of the function $\eta_w(r)$ because the norm of the second fundamental form $\|B^P_x\|$ is bounded by the value of η_w at $r(x)$ instead of $\rho^P(x)$:

Theorem 3.3. *Let $\varphi : P^m \longrightarrow N^n$ be an isometric and proper immersion of a complete non-compact Riemannian m-manifold P^m into a complete Riemannian manifold N^n with a pole $o \in N$ and satisfying $\varphi^{-1}(o) \neq \emptyset$. Let us suppose that, as in Theorem 3.1, the $o-$radial sectional curvatures of N are bounded from above as*

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r(x))}{w(r(x))}\ \ \forall x \in N\,,$$

and for any $r > 0$, $w'(r) \geq d > 0$. Let us assume moreover that the second fundamental form B^P_x in $x \in P$ satisfies that, for sufficiently large radius R_0:

$$\|B^P_x\| \leq c\,\eta_w(r(x))\ \ \forall x \in P - D_{R_0}(o)$$

where c a positive constant such that $c < 1$.

Then P is C^∞- diffeomorphic to the interior of a compact smooth manifold $\overline{P}$ with boundary.

We are going to see how to estimate the area growth function of P, defined as $g(r) = \frac{Vol(\partial D_r)}{Vol(S^w_r)}$ by the number of ends of the immersion P, $\mathcal{E}(P)$, when the ambient space N is a radially symmetric space.

Theorem 3.4. *Let $\varphi : P^m \longrightarrow M^n_w$ be an isometric and proper immersion of a complete non-compact Riemannian m-manifold P^m into a model space M^n_w with pole o_w. Suppose that $\varphi^{-1}(o_w) \neq \emptyset$, $m > 2$ and moreover:*

(1) The norm of second fundamental form B^P_x in $x \in P$ is bounded from above outside a (compact) extrinsic ball $D_{R_0}(o) \subseteq P$ with sufficiently large radius R_0 by:

$$\|B^P_x\|\ \leq\ \frac{\epsilon(r(x))}{(w'(r(x)))^2}\eta_w(r(x))\ \ \forall x \in P - D_{R_0}$$

where ϵ is a positive function such that $\epsilon(r) \to 0$ when $r \to \infty$.

(2) For r sufficiently large, $w'(r) \geq d > 0$.

Then, for sufficiently large r, we have:

$$\frac{Vol(\partial D_r)}{Vol(S^w_r)} \leq \frac{\mathcal{E}(P)}{(1 - 4\epsilon(r))^{\frac{(m-1)}{2}}} \tag{3.1}$$

where $\mathcal{E}(P)$ is the (finite) number of ends of P.

When we consider minimal immersions in the model spaces, we have the following result, which is an inmediate corollary from the above theorem, and Theorem 2.15 in Section 2.

Theorem 3.5. *Let $\varphi : P^m \longrightarrow M^n_w$ be a complete non-compact, proper and minimal immersion into a ballanced from below model space M^n_w with pole o_w. Suppose that $\varphi^{-1}(o_w) \neq \emptyset$ and $m > 2$. Let us assume moreover the hypotheses (1) and (2) in Theorem 3.4.*

Then

(1) *The (finite) number of ends $\mathcal{E}(P)$ is related with the (finite) volume growth by*

$$1 \leq \lim_{r\to\infty} \frac{Vol(D_r)}{Vol(B^w_r)} \leq \mathcal{E}(P) \tag{3.2}$$

(2) *If P has only one end, P is a minimal cone in M^n_w.*

4. Corollaries

As we have said in the Introduction, we have divided the list of results based in Theorem 3.1 and in Theorem 3.4 in two series of corollaries. The first set of consequences follows the line of Theorem 1.1 and Theorem 1.2, (which are in fact the main representatives of these results) presenting upper bounds for the volume and area growth of a complete and proper immersion in the real space form $\mathbb{K}^n(b)$, $(b \leq 0)$, in terms of the number of its ends. In the second set of corollaries, are stated compactification theorems for complete and proper immersions in $\mathbb{R}^n$, $\mathbb{H}^n(b)$ and $\mathbb{H}^n(b) \times \mathbb{R}^l$.

The first of these corollaries constitutes a non-minimal version of Theorem 1.1:

Corollary 4.1. *Let $\varphi : P^m \longrightarrow \mathbb{H}^n(b)$ be a complete non-compact and proper immersion with $m > 2$. Let us suppose that for sufficiently large R_0 and for all points $x \in P$ such that $r(x) > R_0$, (i.e. outside the compact extrinsic ball $D_{R_0}(o)$ with $\varphi^{-1}(o) \neq \emptyset$),*

$$\|B^P_x\| \leq \frac{\delta(r(x))}{e^{2\sqrt{-b}\, r(x)}}$$

where $r(x) = d_{\mathbb{H}^n(b)}(o, \varphi(x))$ is the (extrinsic) distance in $\mathbb{H}^n(b)$ of the points in $\varphi(P)$ to a fixed pole $o \in \mathbb{H}^n(b)$ and $\delta(r)$ is a smooth function such that $\delta(r) \to 0$ when $r \to \infty$.Let $\{t_i\}_{i=1}^{\infty}$ be any non-decreasing sequence such that $t_i \to \infty$ when $i \to \infty$. Then the finite number of ends $\mathcal{E}(P)$ is related with the area growth of P by:

$$\liminf_{i\to\infty} \frac{\mathrm{Vol}(\partial D_{t_i})}{\mathrm{Vol}(S^{b,m-1}_{t_i})} \leq \mathcal{E}(P)$$

The corresponding non-minimal statement of Theorem 1.2 is:

Corollary 4.2. *Let $\varphi : P^m \longrightarrow \mathbb{R}^n$ be a complete non-compact and proper immersion with $m > 2$. Let us suppose that for sufficiently large R_0 and for all points $x \in P$ such that $r(x) > R_0$, (i.e. outside the compact extrinsic ball $D_{R_0}(o)$ with $\varphi^{-1}(o) \neq \emptyset$),*

$$\|B^P_x\| \leq \frac{\epsilon(r(x))}{r(x)}$$

where $r(x) = d_{\mathbb{R}^n}(o, \varphi(x))$ is the (extrinsic) distance in $\mathbb{R}^n$ of the points in $\varphi(P)$ to a fixed pole $o \in \mathbb{R}^n$ and $\epsilon(r)$ is a smooth function such that $\epsilon(r) \to 0$ when $r \to \infty$.

Let $\{t_i\}_{i=1}^{\infty}$ be any non-decreasing sequence such that $t_i \to \infty$ when $i \to \infty$. Then the finite number of ends $\mathcal{E}(P)$ is related with the area growth by:

$$\liminf_{i\to\infty} \frac{\mathrm{Vol}(\partial D_{t_i})}{\mathrm{Vol}(S_{t_i}^{0,m-1})} \leq \mathcal{E}(P)$$

Concerning the compactification results we have the following result given by Bessa, Jorge and Montenegro in [24] and by Bessa and Costa in [25]:

Corollary 4.3. *Let $\varphi : P^m \longrightarrow I\!K^n(b)$ be a complete non-compact immersion in the real space form $I\!K^n(b)$, ($b \leq 0$). Let us suppose that for all points $x \in P \setminus B^P_{R_0}(o)$ (for sufficientlty large R_0, where o is a pole in $I\!K^n(b)$ such that $\varphi^{-1}(o) \neq \emptyset$) :*

$$\|B^P_x\| \leq c\, h_b(\rho^P(x))$$

where $\rho^P(x)$ is the (intrinsic) distance to a fixed $x_o \in \varphi^{-1}(o)$ and c is a positive constant such that $c < 1$ and

$$h_b(r) = \eta_{w_b}(r) = \begin{cases} 1/r & \text{if } b = 0 \\ \sqrt{-b}\coth(\sqrt{-b}\, r) & \text{if } b < 0 \end{cases} .$$

is the mean curvature of the geodesic spheres in $I\!K^n(b)$. Then P is properly immersed in $I\!K^n(b)$ and it is diffeomorphic to the interior of a compact smooth manifold $\overline{P}$ with boundary.

Our last result concerns isometric immersions in $\mathbb{H}^n(b) \times \mathbb{R}^l$:

Corollary 4.4. *Let $\varphi : P^m \longrightarrow \mathbb{H}^n(b) \times \mathbb{R}^l$ be a complete non-compact immersion. Let us consider a pole $o \in \mathbb{H}^n(b) \times \mathbb{R}^l$ such that $\varphi^{-1}(o) \neq \emptyset$. Let us suppose that for all points $x \in P \setminus B^P_{R_0}(x_o)$, where $x_o \in \varphi^{-1}(o)$ and for R_0 sufficiently large:*

$$\|B_x\| \leq \frac{c}{\rho^P(x)} .$$

Here $\rho^P(x)$ denotes the intrinsic distance in P from the fixed $x_o \in \varphi^{-1}(o)$ to x and c is a positive constant such that $c < 1$. Then P is properly immersed in $\mathbb{H}^n(b) \times \mathbb{R}^l$ and it is diffeomorphic to the interior of a compact smooth manifold $\overline{P}$ with boundary.

5. Proof of Theorem 3.1

5.1. P **is properly immersed.** Let us define the following function:

$$F(r) := \int_0^r w(t)dt \tag{5.1}$$

Observe that F is injective, because $F'(r) = w(r) > 0$ $\forall r > 0$, and $F(r) \to \infty$ when $r \to \infty$. Applying Theorem 2.10 and Proposition 2.14, we obtain, for all $x \in P$, and given $X \in T_xP$,

$$\begin{aligned} \mathrm{Hess}^P_x F(r)(X,X) &\geq w'(r(x))\|X\|^2 + w(r(x))\langle B^P_x(X,X), \nabla^N r\rangle \\ &\geq w'(r(x))\|X\|^2 - w(r(x))\|B^P_x\|\,\|X\|^2 \end{aligned} \tag{5.2}$$

By hypotesis there exist a geodesic ball $B^P_{r_1}(x_0)$ in P, with $r_1 \geq R_0$, such that for any $x \in P \setminus B^P_{r_1}(x_0)$, $\|B^P_x\| \leq c\eta_w(\rho^P(x))$. On the other hand, as $\eta_w(r)$ is non-increasing and $r(x) \leq \rho^P(x)$ because φ is isometric, we have $c\eta_w(\rho^P(x)) \leq c\eta_w(r(x))$, so if $x \in P \setminus B^P_{r_1}$:

$$\begin{aligned}\text{Hess}^P_x F(r)(X,X) &\geq w'(r(x))\|X\|^2 - w(r)c\eta_w(\rho^P(x))\,\|X\|^2 \\ &\geq w'(r(x))\|X\|^2\,(1-c) \geq d\,(1-c) > 0\end{aligned} \tag{5.3}$$

The above result implies that there exists $r_1 \geq R_0$ such that $F \circ r$ is a strictly convex function outside the geodesic ball in P centered at x_0, $B^P_{r_1}(x_0)$. And hence, as $r(x) \leq \rho^P(x)$ for all $x \in P$, (and therefore $B^P_{r_1}(x_0) \subseteq D_{r_1}$), $F \circ r$ is a strictly convex function outside the extrinsic disc D_{r_1}.

Let $\sigma : [0, \rho^P(x)] \to P^m$ be a minimizying geodesic from x_0 to x.

If we denote as $f = F \circ r$, let us define $h : \mathbb{R} \to \mathbb{R}$ as

$$h(s) = F(r(\sigma(s))) = f(\sigma(s))$$

Then,

$$(f \circ \sigma)'(s) = h'(s) = \sigma'(s)(f) = \langle \nabla^P f(\sigma(s)), \sigma'(s)\rangle \tag{5.4}$$

and hence,

$$\begin{aligned}(f\circ\sigma)''(s) = h''(s) = \sigma'(s)(\langle \nabla^P f(\sigma(s)), \sigma'(s)\rangle) = \langle \nabla^P_{\sigma'(s)} \nabla^P f(\sigma(s)), \sigma'(s)\rangle \\ + \langle \nabla^P f(\sigma(s)), \nabla^P_{\sigma'(s)}\sigma'(s)\rangle = Hess^P_{\sigma(s)} f(\sigma(s))(\sigma'(s), \sigma'(s))\end{aligned} \tag{5.5}$$

We have from (5.3) that $(f \circ \sigma)''(\tau) = \text{Hess}^P f(\sigma(\tau))(\sigma', \sigma') \geq d(1-c)$ for all $\tau \geq r_1$. And for $\tau < r_1$, $(f \circ \sigma)''(\tau)) \geq a = \inf_{x \in B^P_{r_1}} \{\text{Hess}^P f(x)(\nu, \nu),\, |\nu| = 1\}$. Then

$$\begin{aligned}(f\circ\sigma)'(s) &= (f\circ\sigma)'(0) + \int_0^s (f\circ\sigma)''(\tau)d\tau \\ &\geq (f\circ\sigma)'(0) + \int_0^{r_1} a\,d\tau + d\int_{r_1}^s (1-c)d\tau \\ &\geq (f\circ\sigma)'(0) + a\,r_1 + d\,(1-c)(s-r_1)\end{aligned} \tag{5.6}$$

On the other hand, as

$$\nabla^P f(\sigma(s)) = \nabla^P F(r(\sigma(s))) = F'(r(\sigma(s)))\nabla^P r|_{\sigma(s)} = w(r(\sigma(s)))\nabla^P r|_{\sigma(s)} \tag{5.7}$$

then

$$\nabla^P f(\sigma(0)) = w(r(\sigma(0)))\nabla^P r|_{\sigma(0)} = w(0)\nabla^P r|_{\sigma(0)} = 0$$

so we have that

$$(f\circ\sigma)'(0) = \langle \nabla^P f(\sigma(0)), \sigma'(0)\rangle = 0 \tag{5.8}$$

We also have that $(f\circ\sigma)(0) = F(r(\sigma(0))) = F(0) = 0$. Hence, applying inequality (5.6),

$$f(\sigma(s)) = (f\circ\sigma)(0) + \int_0^s (f\circ\sigma)'(\tau)d\tau \geq ar_1 s + d(1-c)\{\frac{1}{2}s^2 - r_1 s\} \tag{5.9}$$

Therefore,

$$(5.10)\quad \begin{aligned} F(r(x)) &= f(x) = f(\sigma(\rho^P(x))) = \int_0^{\rho^P(x)} (f\circ\sigma)'(s)\,ds \\ &\geq \int_0^{\rho^P(x)} a\,r_1 + d\,(1-c)(s-r_1)\,ds \\ &= a\,r_1\rho^P(x)\ + d\,(1-c)\left(\frac{\rho^P(x)^2}{2} - r_1\,\rho^P(x)\right) \end{aligned}$$

Hence, if $\rho^P \to \infty$ then $F(r(x)) \to \infty$ and then, as F is strictly increasing, $r \to \infty$ so the immersion is proper.

5.2. P **has finite topology.** We are going to see that $\nabla^P r$ never vanishes on $P \setminus D_{r_1}$. To show this, we consider, as in the previous subsection, any geodesic in P emanating from the pole o, $\sigma(s)$. We have, using inequality (5.6), that

$$(5.11)\quad \langle \nabla^P f(\sigma(s)), \sigma'(s)\rangle = (f\circ\sigma)'(s) \geq a\,r_1 + d\,(1-c)(s-r_1) > 0\ \forall s > r_1$$

Hence, as $\|\sigma'(s)\| = 1\ \forall s$, then $\|\nabla^P f(\sigma(s))\| > 0$ for all $s > r_1$. But we have computed $\nabla^P f(\sigma(s)) = w(r(\sigma(s)))\nabla^P r|_{\sigma(s)}$, so, as $w(r) > 0\ \forall r > 0$, then $\|\nabla^P r|_{\sigma(s)}\| > 0\ \forall s > r_1$ and hence, $\nabla^P r|_{\sigma(s)} \neq 0\ \forall s > r_1$. We have proved that $\nabla^P r$ never vanishes on $P \setminus B^P_{r_1}$, so we have too that $\nabla^P r$ never vanishes on $P \setminus D_{r_1}$. Let

$$\phi : \partial D_{r_1} \times [r_1, +\infty) \to P \setminus D_{r_1}$$

be the integral flow of a vector field $\frac{\nabla^P r}{\|\nabla^P r\|^2}$ with

$$\phi(p, r_1) = p \in \partial D_{r_1}$$

It is obvious that $r(\phi(p,t)) = t$ and

$$\phi(\cdot, t) : \partial D_{r_1} \to \partial D_t$$

is a diffeomorphism. So P has finitely many ends, and each of its ends is of finite topological type.

In fact, applying Theorem 3.1 in [20], we conclude that, as the extrinsic annuli $A_{r_1,R}(o) = D_R(o) \setminus D_{r_1}(o)$ contain no critical points of the extrinsic distance function $r : P \longrightarrow \mathbb{R}^+$, then $D_R(o)$ is diffeomorphic to $D_{r_1}(o)$ for all $R \geq r_1$ and hence the annuli $A_{r_1,R}(o)$ are diffeomorphic to $\partial D_{r_1} \times [r_1, R]$.

Remark 5.1. To show Theorem 3.3, we argue as in the beginning of the proof of Theorem 3.1: with the same function $F(r)$ we obtain inequality (5.2). But now we have as hypothesis that $\|B^P_x\| \leq c\,\eta_w(r(x))$, so we don't need that $\eta'_w(r) \leq 0$ to get inequality (5.3).

6. Proof of Theorem 3.4

We are going to see first that P has finite topology. As P is properly immersed, we shall apply Theorem 3.3 and for that, it must be checked that hypotheses in that theorem are acomplished. First, we have hypothesis (1) in Theorem 3.3 because $N = M^n_w$. On the other hand, as $w'(r) \geq d > 0 \forall r > 0$ and, for some R_0, we have that $\|B^P_x\| \leq \frac{\epsilon(r(x))}{(w'(r(x)))^2}\eta_w(r(x))\ \ \forall x \in P - D_{R_0}$ where ϵ is a positive function such that $\epsilon(r) \to 0$ when $r \to \infty$, hence $0 \leq \lim_{r\to\infty} \frac{\epsilon(r)}{(w'(r))^2} \leq \lim_{r\to\infty} \frac{\epsilon(r)}{d^2} = 0$. Therefore, for some constant $c < 1$, there exist R_0 such that $\|B^P_x\| \leq c\eta_w(r(x))$ $\forall x \in P - D_{R_0}$. Therefore, as $\varphi : P \longrightarrow M^n_w$ is a proper immersion, we have by Theorem 3.3 that P has finite topological type and thus P has finitely many ends, each of finite topological type. Hence we have, in an analogous way than in [1], and for $r_1 \geq R_0$ as in Section 5:

$$P - D_{r_1} = \cup_{k=1}^{\mathcal{E}(P)} V_k \tag{6.1}$$

where V_k are disjoint, smooth domains in P. Along the rest of the proof, we will work on each end V_k separately. Let V denote one element of the family $\{V_k\}_{k=1}^{\mathcal{E}(P)}$, and, given a fixed radius $t > r_1$, let $\partial V(t)$ denote the set $\partial V(t) = V \cap \partial D_t = V \cap S^w_t$, where S^w_t is the geodesic t-sphere in M^n_w. This set is a hypersurface in P^m, with normal vector $\frac{\nabla^P r}{\|\nabla^P r\|}$, and we are going to estimate its sectional curvatures when $t \to \infty$.

Suppose that e_i, e_j are two orthonormal vectors of $T_p\partial V(t)$ on the point $p \in \partial V(t)$. Then the sectional curvature of the plane expanded by e_i, e_j is, using Gauss formula:

$$\begin{aligned}
K_{\partial V(t)}(e_i, e_j) &= K_P(e_i, e_j) + \langle B^{\partial V-P}(e_i, e_i), B^{\partial V-P}(e_j, e_j)\rangle \\
&- \|B^{\partial V-P}(e_i, e_j)\|^2 = K_N(e_i, e_j) + \langle B^{\partial V-P}(e_i, e_i), B^{\partial V-P}(e_j, e_j)\rangle \\
&- \|B^{\partial V-P}(e_i, e_j)\|^2 + \langle B^P(e_i, e_i), B^P(e_j, e_j)\rangle - \|B^P(e_i, e_j)\|^2 \\
&\geq K_N(e_i, e_j) + \langle B^{\partial V-P}(e_i, e_i), B^{\partial V-P}(e_j, e_j)\rangle \\
&- \|B^{\partial V-P}(e_i, e_j)\|^2 - 2\|B^P\|^2
\end{aligned} \tag{6.2}$$

where $B^{\partial V-P}$ is the second fundamental form of $\partial V(t)$ in P. But this second fundamental form is for two vector fields X, Y in $T\partial V(t)$:

$$\begin{aligned}
B^{\partial V-P}(X, Y) &= \langle \nabla^P_X Y, \frac{\nabla^P r}{||\nabla^P r||}\rangle \frac{\nabla^P r}{||\nabla^P r||} = \langle \nabla^P_X Y, \nabla^P r\rangle \frac{\nabla^P r}{||\nabla^P r||^2} \\
&= X(\langle Y, \nabla^P r\rangle)\frac{\nabla^P r}{||\nabla^P r||^2} - \langle Y, \nabla^P_X \nabla^P r\rangle \frac{\nabla^P r}{||\nabla^P r||^2} \\
&= -\text{Hess}^P r(X, Y)\frac{\nabla^P r}{||\nabla^P r||^2}
\end{aligned} \tag{6.3}$$

Then, since, for all $X, Y \in T_pM^n_w$

$$\text{Hess}^{M^n_w} r(X, Y) = \eta_w(r)\langle X, Y\rangle - \langle X, \nabla^{M^n_w} r\rangle\langle Y, \nabla^{M^n_w} r\rangle \tag{6.4}$$

we have, (using the fact that e_i are tangent to the fiber S^w_t, and Proposition 2.6), that

$$K_{M^n_w}(e_i, e_j) = K(t) = \frac{1}{w^2(t)} - \eta^2_w(t) \tag{6.5}$$

so for any $p \in \partial V(t)$ such that $t = r(p)$ is sufficiently large:

$$
\begin{aligned}
K_{\partial V(t)}(e_i,e_j) \geq & K_{M_w^n}(e_i,e_j) + \frac{\operatorname{Hess}_p^P r(e_i,e_i)\operatorname{Hess}_p^P r(e_j,e_j)}{||\nabla^P r||^2} \\
& - \frac{\operatorname{Hess}_p^P r(e_i,e_j)^2}{||\nabla^P r||^2} - 2\|B^P\|^2 \\
& \geq K(t) + \frac{\left(\eta_w(t) - \|B^P\|\right)^2 - \|B^P\|^2}{||\nabla^P r||^2} - 2\|B^P\|^2 \\
& \geq \eta_w^2(t)\left(1 - 2\frac{\|B^P\|}{\eta_w(t)} - 2\left(\frac{\|B^P\|}{\eta_w(t)}\right)^2 + \frac{K(t)}{\eta_w^2(t)}\right) \qquad (6.6) \\
& \geq \eta_w^2(t)\left(1 - 4\frac{\|B^P\|}{\eta_w(t)} + \frac{K(t)}{\eta_w^2(t)}\right) \\
& = \eta_w^2(t)\left(1 + \frac{K(t)}{\eta_w^2(t)}\right)\left(1 - 4\frac{\frac{\|B^P\|}{\eta_w(t)}}{1 + \frac{K(t)}{\eta_w^2(t)}}\right) \\
& \geq \frac{1}{w^2(t)}\left(1 - 4\|B^P\|w'(t)w(t)\right) \geq \frac{1}{w^2(t)}\left(1 - 4\epsilon(t)\right)
\end{aligned}
$$

where we recall that, by hypothesis, $\|B^P\| \leq \frac{\epsilon(t)}{(w'(t))^2}\eta_w(t)$ for all $t = r(x) > R_0$, and ϵ is a positive function such that $\epsilon(r) \to 0$ when $r \to \infty$.

If we denote as $\delta(t) = \frac{1}{w^2(t)}\left(1 - 4\epsilon(t)\right)$ we have for each t sufficiently large that $K_{\partial V(t)}(e_i,e_j) \geq \delta(t)$ holds everywhere on $\partial V(t)$ and $\delta(t)$ is a positive constant. Then, the Ricci curvature of $\partial V(t)$ is bounded from below, for these sufficiently large radius t as

$$
Ricc_{\partial V(t)}(\xi,\xi) \geq \delta(t)(m-1)\|\xi\|^2 > 0 \ \forall \xi \in T\partial V(t)
$$

so, applying Myers' Theorem $\partial V(t)$ is compact and has diameter $d(\partial V(t)) \leq \frac{\pi}{\sqrt{\delta(t)}}$ (see [30]). Applying on the other hand Bishop's Theorem, (see Theorem 6 in [2]), we obtain:

$$
\operatorname{Vol}(\partial V(t)) \leq \frac{\operatorname{Vol}(S^{0,m-1}(1))}{\sqrt{\delta(t)^{m-1}}} \qquad (6.7)
$$

and hence

$$
\begin{aligned}
\frac{\operatorname{Vol}(\partial V(t))}{\operatorname{Vol}(S_t^w)} \leq & \frac{1}{w(t)^{m-1}\sqrt{\delta(t)^{m-1}}} \qquad (6.8) \\
& = \frac{1}{\left(1 - 4\epsilon(t)\right)^{(m-1)/2}}
\end{aligned}
$$

Therefore, since for t large enough $Vol(\partial D_t(o)) \leq \sum_{i=1}^{\mathcal{E}(P)} Vol(\partial V_i(t))$ where V_i denotes each end of P then:

$$
\frac{\operatorname{Vol}(\partial D_t(o))}{\operatorname{Vol}(S_t^w)} \leq \frac{\mathcal{E}(P)}{\left(1 - 4\epsilon(t)\right)^{(m-1)/2}} \qquad (6.9)
$$

7. Proof of Theorem 3.5

To show assertion (1) we apply Theorem 2.15 and inequality (3.1) in Theorem 3.4 to obtain, for r sufficiently large, (we suppose that $\varphi^{-1}(o_w) \neq \emptyset$, and take $o \in \varphi^{-1}(o_w)$ in order to have that $\mathrm{Vol}(D_r(o)) \geq \mathrm{Vol}(B_r^w)$ for all $r > 0$) :

$$\begin{aligned} 1 \leq & \frac{\mathrm{Vol}(D_r(o))}{\mathrm{Vol}(B_r^w)} \leq \frac{\mathrm{Vol}(\partial D_r(o))}{\mathrm{Vol}(S_r^w)} \\ & \leq \frac{\mathcal{E}(P)}{(1-4\epsilon(r))^{(m-1)/2}} \end{aligned} \tag{7.1}$$

Moreover, we know (again using Theorem 2.15) that the volume growth function is non-decreasing.

Therefore, taking limits in (7.1) when r goes to ∞, we obtain:

$$1 \leq \lim_{r\to\infty} \frac{\mathrm{Vol}(D_r(o))}{\mathrm{Vol}(B_r^w)} = \mathrm{Sup}_{r>0} \frac{\mathrm{Vol}(D_r(o))}{\mathrm{Vol}(B_r^w)} \leq \mathcal{E}(P) \tag{7.2}$$

Now, to prove assertion (2), we have, if P has one end, that

$$1 \leq \mathrm{Sup}_{r>0} \frac{\mathrm{Vol}(D_r(o))}{\mathrm{Vol}(B_r^w)} \leq 1 \tag{7.3}$$

Hence, as $f(r) = \frac{\mathrm{Vol}(D_r(o))}{\mathrm{Vol}(B_r^w)}$ is non-decreasing, then $f(r) = 1\ \forall r > 0$, so we have equality in inequality (2.6) for all $r > 0$, and P is a minimal cone, (see [17] for details).

8. Proof of Theorems 1.1 and 1.2 and the Corollaries

8.1. **Proof of Theorem 1.1.** We are going to apply Theorem 3.5. To do that, we must to check hypotheses (1) and (2) in Theorem 3.4.

We have, in this case, that the ambient manifold is the hyperbolic space $\mathbb{H}^n(b)$. Therefore all of its points are poles, so there exist at least $o \in \mathbb{H}^n(b)$ such that $\varphi^{-1}(o) \neq \emptyset$. As it is known, Hyperbolic space $\mathbb{H}^n(b)$ is a model space with $w(r) = w_b(r) = \frac{1}{\sqrt{-b}} \sinh\sqrt{-b}r$ so $w_b'(r) = \cosh\sqrt{-b}r \geq 1\ \forall r > 0$.

Therefore, hypothesis (2) in Theorem 3.4 is fulfilled in this context. Concerning hypothesis (1), it is straightforward that

$$\begin{aligned} \|B_x^P\| \leq & \frac{\delta(r(x))}{e^{2\sqrt{-b}\,r(x)}} \leq \frac{\epsilon(r)\sqrt{-b}}{\sinh\sqrt{-b}r\cosh\sqrt{-b}r} \\ & = \frac{\epsilon(r)}{\cosh^2\sqrt{-b}r}\sqrt{-b}\coth\sqrt{-b}r = \frac{\epsilon(r)}{(w_b'(r))^2}\eta_{w_b}(r) \end{aligned} \tag{8.1}$$

where $\epsilon(r) = \frac{\delta(r(x))}{4\sqrt{-b}}$ goes to 0 when r goes to ∞.

Hence, also hypothesis (1) in Theorem 3.4 is fulfilled so, applying inequality (3.2) in Theorem 3.5, (because P is minimal)

$$1 \leq \lim_{r\to\infty} \frac{Vol(D_r)}{Vol(B_r^{w_b})} \leq \mathcal{E}(P) \tag{8.2}$$

Finally, when P has one end, then $\lim_{r\to\infty} \frac{Vol(D_r)}{Vol(B_r^{w_b})} = 1$. Since P is minimal, by Theorem 2.15, $f(r) = \frac{Vol(D_r)}{Vol(B_r^{w_b})}$ is a monotone non-decreasing function, and, on the other hand, $f(r) \geq 1$ $\forall r > 0$ because inequality (2.7). Hence $f(r) = 1$ $\forall r > 0$, so $f'(r) = 0$ $\forall r > 0$. This last equality implies the equality in inequality (2.6) for all $r > 0$, (see [17] or [18] for details), and we apply equality assertion in Theorem 2.15 to conclude that P is totally geodesic in $\mathbb{H}^n(b)$.

8.2. **Proof of Theorem 1.2.** In this case, we apply Theorem 3.5, being $M_w^n = \mathbb{R}^n$, i.e., being $w(r) = w_0(r) = r$, $(b = 0)$. Hence, $w_0'(r) = 1 > 0$ $\forall r > 0$ and $\eta_0(r) = \frac{1}{r}$ and hypotheses (1) and (2) in this theorem are trivially satisfied.

When P has only one end we conclude as before that the volume growth function is constant so we conclude equality in (2.6) for all radius $r > 0$. Hence P is totally geodesic in $\mathbb{R}^n$ applying the corresponding equality assertion in Theorem 2.15.

8.3. **Proof of Corollary 4.1.** We are considering now a complete and proper immersion in $\mathbb{H}^n(b)$, as in Theorem 1.1, but P is not necessarily minimal. In this setting hypotheses (1) and (2) in Theorem 3.4 are fulfilled (as we have checked in the proof above, without using minimality). Hence taking limits in (3.1) when we consider an increasing sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_i \to \infty$ when $i \to \infty$, we have:

$$\liminf_{i\to\infty} \frac{\operatorname{Vol}(\partial D_{t_i})}{\operatorname{Vol}(S_{t_i}^{b,m-1})} \leq \mathcal{E}(P)$$

8.4. **Proof of Corollary 4.2.** Hypotheses (1) and (2) in Theorem 3.4 are trivially satisfied and we argue as in the proof of Corollary 4.1 to obtain the result.

8.5. **Proof of Corollary 4.3.** We apply Theorem 3.1. Our ambient manifold is $\mathbb{K}^n(b)$, $(b \leq 0)$, so hypothesis (1) about the bounds for the radial sectional curvature holds, and as $w(r) = w_b(r)$ hence $w_b'(r) \geq 1 > 0$ $\forall r > 0$ and $\eta_{w_b}'(r) \leq 0$ $\forall r > 0$. This means that hypothesis (3) is fulfilled. Hypothesis (2) in Theorem 3.1 holds because

$$\|B_x^P\| \leq c\, h_b(\rho^P(x))$$

where $\rho^P(x)$ is the (intrinsic) distance to a fixed $x_o \in \varphi^{-1}(o)$ and c is a positive constant such that $c < 1$.

8.6. **Proof of Corollary 4.4.** We apply again Theorem 3.1, having into account that the ambient space is the Cartan-Hadamard manifold $\mathbb{H}^n(b) \times \mathbb{R}^l$ and the model space used to compare is $\mathbb{R}^m$, with $w(r) = w_0(r) = r$.

References

[1] M. Anderson *The compactification of a minimal submanifold by the Gauss Map.*, Preprint IEHS (1984).

[2] I. Chavel *Eigenvalues in Riemannian geometry*, vol. 115 of *Pure and Applied Mathematics*, Academic Press Inc., Orlando, FL, 1984. Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk.

[3] Q. Chen *On the volume growth and the topology of complete minimal submanifolds of a Euclidean space* J. Math. Sci. Univ. Tokyo 2 (1995), 657-669.

[4] S. S. Chern & R. Osserman *Complete minimal surfaces in euclidean n-space.*, J. d'Analyse Math. vol. 19, 15-34, (1967).

[5] M. P. do Carmo *Riemannian Geometry*, Birkhauser, Boston Inc., 1992.

[6] R. Greene and H. Wu *Function theory on manifolds which possess a pole.*, Lecture Notes in Math., vol. 699, Springer-Verlag, Berlin and New York, 1979.

[7] R. Greene and H. Wu *Gap theorems for noncompact Riemannian manifolds*, Duke Math. J. 49 (1982), 731-756.

[8] R. Greene and H. Wu, *On a new gap phenomenon in riemannian geometry*, Proc. Nac. Acad. Sci. USA. 79 (1982), 714-715.

[9] R. E. Greene, P. Petersen & S. Zhu *Riemannian Manifolds of Faster-Than-Quadratic Curvature Decay*, Int. Math. Research Notices, 1994, No. 9.

[10] A. Grigor'yan *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*, Bull. Amer. Math. Soc. **36** (1999), 135–249.

[11] Luquésio P. Jorge & W. Meeks III *The topology of complete minimal surfaces of finite total Gaussian curvature.*, Topology, vol. 22 (2), 203-221, (1983).

[12] A. Kasue, & K. Sugahara *Gap theorems for certain submanifolds of Euclidean spaces and hyperbolic space forms.*, Osaka J. Math. 24 (1987), 679-704.

[13] A. Kasue *Gap theorems for minimal submanifolds of Euclidean space.*, J. Math. Soc. Japan 38 (3) (1986), 473-492.

[14] H. B. Lawson *Lectures on minimal submanifolds.*, Monografias de Matemática, IMPA, Rio de Janeiro, Brazil.

[15] S. Markvorsen *On the mean exit time from a minimal submanifold.*, J. Diff. Geom. **29** (1989), 1–8

[16] S. Markvorsen & V. Palmer *Torsional rigidity of minimal submanifolds.*, Proc. London Math Soc.(3) 93 (2006) 253-272.

[17] S. Markvorsen and V. Palmer *The relative volume growth of minimal submanifolds.*, Archiv der Mathematik, 79 (2002), 507–514.

[18] S. Markvorsen and V. Palmer *Generalized isoperimetric inequalities for extrinsic balls in minimal submanifolds.*, Journal reine angew. Math., 551 (2002), 101–121.

[19] S. Markvorsen and V. Palmer *On the isoperimetric rigidity of extrinsic minimal balls.*, Differential Geometry and its Applications , 18 (2003), 47–54.

[20] J. Milnor *Morse theory*, Lecture Notes in Math., 699, (1979), Springer Verlag, Berlin.

[21] S. Muller & V. Sverak *On surfaces of finite total curvature.*, J. Diff. Geometry, 42, 229-258 (1995).

[22] G. De Oliveira *Compactification of minimal submanifolds of hyperbolic space.*, Comm. Analysis and Geometry, 1 (1993), 1-29.

[23] B. O'Neill *Semi-Riemannian Geometry; With Applications to Relativity.*, Academic Press (1983).

[24] G. Pacelli Bessa & L. Jorge & J. Fabio Montenegro *Complete submanifolds of* $\mathbb{R}^n$ *with finite topology.*, Communications in Analysis and Geometry, ISSN 1019-8385, Vol. 15, 4, 2007 , 725-732

[25] G. Pacelli Bessa & M. Silvana Costa .*On Submanifolds With Tamed Second Fundamental Form.*, Glasgow Mathematical Journal, 51, 2009, 669-680

[26] V. Palmer *Isoperimetric inequalities for extrinsic balls in minimal submanifolds and their applications.*, J. London Math. Soc. (2) 60, 2 (1999), 607–616.

[27] V. Palmer *On deciding whether a submanifold is parabolic of hyperbolic using its mean curvature* , Simon Stevin Transactions on Geometry, vol 1. 131-159, Simon Stevin Institute for Geometry, Tilburg, The Netherlands, 2010.

[28] R. Osserman *Global properties of minimal surfaces in* $\mathbb{E}^3$ *and* $\mathbb{E}^n$., Ann. of Math. vol. 80, 340-364, (1964).

[29] Q.H. Ruan *New gap theorem on complete Riemannian manifolds.*, (2006). Retrieved from http://arxiv.org/abs/math/0605360

[30] T. Sakai *Riemannian Geometry*, Translations of Mathematical Monographs, vol. 149, A.M.S.1996Addison-Wesley, Reading, MA 1990.

[31] K. Shiohama *Total curvature and minimal areas of complete open surfaces.*, Proc. Amer. Math. Soc, 94 num 2 (1985), 310-316.

[32] R. Schoen *Uniqueness, symmetry and embededness of minimal surfaces.*, J. Diff. Geometry, 18 (1983), 791-809.

[33] Y.T. Siu & S.T. Yau *Complete Kähler manifolds with nonpositive curvature of faster than quadratic decay.*, Annals of Math, 105 (1977), 225-264.

[34] B. White *Complete surfaces of finite total curvature.*, J. Diff. Geometry, 26, 315-326 (1987)

Departament de Matemàtiques- Institut of New Imaging Technologies, Universitat Jaume I, Castellon, Spain.

E-mail address: gimenov@uji.es

Departament de Matemàtiques- Institut of New Imaging Technologies, Universitat Jaume I, Castellon, Spain.

E-mail address: palmer@mat.uji.es

Bibliografia

[And82] Michael T. Anderson, *Complete minimal varieties in hyperbolic space*, Invent. Math. **69** (1982), no. 3, 477–494. MR 679768 (84c:53005)

[And84] ______, *The compactification of a minimal submanifold in euclidean space by the gauss map*, 1984.

[BC64] Richard L. Bishop and Richard J. Crittenden, *Geometry of manifolds*, Pure and Applied Mathematics, Vol. XV, Academic Press, New York, 1964. MR 0169148 (29 #6401)

[CGT82] Jeff Cheeger, Mikhail Gromov, and Michael Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*, J. Differential Geom. **17** (1982), no. 1, 15–53. MR 658471 (84b:58109)

[Cha84] Isaac Chavel, *Eigenvalues in Riemannian geometry*, Pure and Applied Mathematics, vol. 115, Academic Press Inc., Orlando, FL, 1984, Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk. MR 768584 (86g:58140)

[Cha93] ______, *Riemannian geometry—a modern introduction*, Cambridge Tracts in Mathematics, vol. 108, Cambridge University Press, Cambridge, 1993. MR 1271141 (95j:53001)

[Cha01] ______, *Isoperimetric inequalities*, Cambridge Tracts in Mathematics, vol. 145, Cambridge University Press, Cambridge, 2001, Differential geometric and analytic perspectives. MR 1849187 (2002h:58040)

[Cheeger70] Jeff Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, Problems in analysis (Papers dedicated to Salomon Bochner, 1969), Princeton Univ. Press, Princeton, N. J., 1970, pp. 195–199. MR 0402831 (53 #6645)

[Che84] Bang-Yen Chen, *Total mean curvature and submanifolds of finite type*, Series in Pure Mathematics, vol. 1, World Scientific Publishing Co., Singapore, 1984. MR 749575 (86b:53053)

[CheQi95] Qing Chen, *On the volume growth and the topology of complete minimal submanifolds of a Euclidean space*, J. Math. Sci. Univ. Tokyo **2** (1995), no. 3, 657–669. MR 1382525 (97g:53074)

[CheQi97] ______, *On the total curvature and area growth of minimal surfaces in* $\mathbf{R}^n$, Manuscripta Math. **92** (1997), no. 2, 135–142. MR 1428644 (98c:49073)

[CO67] Shiing-shen Chern and Robert Osserman, *Complete minimal surfaces in euclidean n-space*, J. Analyse Math. **19** (1967), 15–34. MR 0226514 (37 #2103)

[CO84] ______, *Complete minimal surfaces in euclidean space*, Academic Press, New York, 1984.

[dC92] Manfredo Perdigão do Carmo, *Riemannian geometry*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1992, Translated from the second Portuguese edition by Francis Flaherty. MR 1138207 (92i:53001)

[dC94] Manfredo P. do Carmo, *Differential forms and applications*, Universitext, Springer-Verlag, Berlin, 1994, Translated from the 1971 Portuguese original. MR 1301070 (95j:58002)

[dOF93] Geraldo de Oliveira Filho, *Compactification of minimal submanifolds of hyperbolic space*, Comm. Anal. Geom. **1** (1993), no. 1, 1–29. MR 1230271 (94h:53080)

[GG73] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Springer-Verlag, New York, 1973, Graduate Texts in Mathematics, Vol. 14. MR 0341518 (49 #6269)

[GPZ94] Robert E. Greene, Peter Petersen, and Shun-Hui Zhu, *Riemannian manifolds of faster-than-quadratic curvature decay*, Internat. Math. Res. Notices (1994), no. 9, 363ff., approx. 16 pp. (electronic). MR 1301436 (95m:53054)

[Gra04] Alfred Gray, *Tubes*, second ed., Progress in Mathematics, vol. 221, Birkhäuser Verlag, Basel, 2004, With a preface by Vicente Miquel. MR 2024928 (2004j:53001)

[Gri99] Alexander Grigor′yan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*, Bull. Amer. Math. Soc. (N.S.) **36** (1999), no. 2, 135–249. MR 1659871 (99k:58195)

[Gri09] Alexander Grigor'yan, *Heat kernel and analysis on manifolds*, AMS/IP Studies in Advanced Mathematics, vol. 47, American Mathematical Society, Providence, RI, 2009. MR 2569498 (2011e:58041)

[Gün60] Paul Günther, *Einige Sätze über das Volumenelement eines Riemannschen Raumes*, Publ. Math. Debrecen **7** (1960), 78–93. MR 0141058 (25 #4471)

[GW79] R. E. Greene and H. Wu, *Function theory on manifolds which possess a pole*, Lecture Notes in Mathematics, vol. 699, Springer, Berlin, 1979. MR 521983 (81a:53002)

[GW82a] ______, *Gap theorems for noncompact Riemannian manifolds*, Duke Math. J. **49** (1982), no. 3, 731–756. MR 672504 (84m:53050)

[GW82b] ______, *On a new gap phenomenon in Riemannian geometry*, Proc. Nat. Acad. Sci. U.S.A. **79** (1982), no. 2, 714–715. MR 648065 (83b:53040)

[HMP09] A. Hurtado, S. Markvorsen, and V. Palmer, *Torsional rigidity of submanifolds with controlled geometry*, Math. Ann. **344** (2009), no. 3, 511–542. MR 2501301 (2010g:53108)

[HP11] Ana Hurtado and Vicente Palmer, *A note on the p-parabolicity of submanifolds*, Potential Anal. **34** (2011), no. 2, 101–118. MR 2754966 (2012e:53099)

[HS74] David Hoffman and Joel Spruck, *Sobolev and isoperimetric inequalities for Riemannian submanifolds*, Comm. Pure Appl. Math. **27** (1974), 715–727. MR 0365424 (51 #1676)

[JK81] L. Jorge and D. Koutroufiotis, *An estimate for the curvature of bounded submanifolds*, Amer. J. Math. **103** (1981), no. 4, 711–725. MR 623135 (83d:53041b)

[JM83] Luquésio P. Jorge and William H. Meeks, III, *The topology of complete minimal surfaces of finite total Gaussian curvature*, Topology **22** (1983), no. 2, 203–221. MR 683761 (84d:53006)

[Jos02] Jürgen Jost, *Riemannian geometry and geometric analysis*, third ed., Universitext, Springer-Verlag, Berlin, 2002. MR 1871261 (2002i:53001)

[Kas86] Atsushi Kasue, *Gap theorems for minimal submanifolds of Euclidean space*, J. Math. Soc. Japan **38** (1986), no. 3, 473–492. MR 845714 (87h:53084)

[KN96] Shoshichi Kobayashi and Katsumi Nomizu, *Foundations of differential geometry. Vol. II*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1996, Reprint of the 1969 original, A Wiley-Interscience Publication. MR 1393941 (97c:53001b)

[KS87] Atsushi Kasue and Kunio Sugahara, *Gap theorems for certain submanifolds of Euclidean spaces and hyperbolic space forms*, Osaka J. Math. **24** (1987), no. 4, 679–704. MR 927056 (89a:53072)

[Law80] H. Blaine Lawson, Jr., *Lectures on minimal submanifolds. Vol. I*, second ed., Mathematics Lecture Series, vol. 9, Publish or Perish Inc., Wilmington, Del., 1980. MR 576752 (82d:53035b)

[Mar89] Steen Markvorsen, *On the mean exit time from a minimal submanifold*, J. Differential Geom. **29** (1989), no. 1, 1–8. MR 978073 (90a:58190)

[Mas77] William S. Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York, 1977, Reprint of the 1967 edition, Graduate Texts in Mathematics, Vol. 56. MR 0448331 (56 #6638)

[Mil63] J. Milnor, *Morse theory*, Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51, Princeton University Press, Princeton, N.J., 1963. MR 0163331 (29 #634)

[MP02a] Steen Markvorsen and Vicente Palmer, *Generalized isoperimetric inequalities for extrinsic balls in minimal submanifolds*, J. Reine Angew. Math. **551** (2002), 101–121. MR 1932175 (2003k:53072)

[MP02b] ______, *The relative volume growth of minimal submanifolds*, Arch. Math. (Basel) **79** (2002), no. 6, 507–514. MR 1967269 (2004a:53078)

[MP03] ______, *On the isoperimetric rigidity of extrinsic minimal balls*, Differential Geom. Appl. **18** (2003), no. 1, 47–54. MR 1951079 (2003k:53046)

[MP06] ______, *Torsional rigidity of minimal submanifolds*, Proc. London Math. Soc. (3) **93** (2006), no. 1, 253–272. MR 2235949 (2008a:53060)

[MP10] ______, *Extrinsic isoperimetric analysis of submanifolds with curvatures bounded from below*, J. Geom. Anal. **20** (2010), no. 2, 388–421. MR 2579515 (2011c:53131)

[MP11] William H. Meeks, III and Joaquín Pérez, *The classical theory of minimal surfaces*, Bull. Amer. Math. Soc. (N.S.) **48** (2011), no. 3, 325–407. MR 2801776

[MS86] Vitali D. Milman and Gideon Schechtman, *Asymptotic theory of finite-dimensional normed spaces*, Lecture Notes in Mathematics, vol. 1200, Springer-Verlag, Berlin, 1986, With an appendix by M. Gromov. MR 856576 (87m:46038)

[MŠ95] S. Müller and V. Šverák, *On surfaces of finite total curvature*, J. Differential Geom. **42** (1995), no. 2, 229–258. MR 1366547 (97b:53007)

[O'N83] Barrett O'Neill, *Semi-Riemannian geometry*, Pure and Applied Mathematics, vol. 103, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1983, With applications to relativity. MR 719023 (85f:53002)

[Oss64] Robert Osserman, *Global properties of minimal surfaces in E^3 and E^n*, Ann. of Math. (2) **80** (1964), 340–364. MR 0179701 (31 #3946)

[Pal99] Vicente Palmer, *Isoperimetric inequalities for extrinsic balls in minimal submanifolds and their applications*, J. London Math. Soc. (2) **60** (1999), no. 2, 607–616. MR 1724821 (2000j:53050)

[Pal10] ______, *On deciding whether a submanifold is parabolic of hyperbolic using its mean curvature*, Simon Stevin Institute for Geometry **1** (2010), 131–159.

[PBC09] G. Pacelli Bessa and M. Silvana Costa, *On submanifolds with tamed second fundamental form*, Glasg. Math. J. **51** (2009), no. 3, 669–680. MR 2534016 (2010h:53088)

[PBJM07] G. Pacelli Bessa, Luquésio Jorge, and J. Fabio Montenegro, *Complete submanifolds of $\mathbb{R}^n$ with finite topology*, Comm. Anal. Geom. **15** (2007), no. 4, 725–732. MR 2395255 (2009d:53079)

[Pet98] Peter Petersen, *Riemannian geometry*, Graduate Texts in Mathematics, vol. 171, Springer-Verlag, New York, 1998. MR 1480173 (98m:53001)

[Rua06] Q.H. Ruan, *New gap theorem on complete riemannian manifolds*, http://arxiv.org/abs/math/0605360 (2006).

[Sak96] Takashi Sakai, *Riemannian geometry*, Translations of Mathematical Monographs, vol. 149, American Mathematical Society, Providence, RI, 1996, Translated from the 1992 Japanese original by the author. MR 1390760 (97f:53001)

[Sch83] Richard M. Schoen, *Uniqueness, symmetry, and embeddedness of minimal surfaces*, J. Differential Geom. **18** (1983), no. 4, 791–809 (1984). MR 730928 (85f:53011)

[Shi85] Katsuhiro Shiohama, *Total curvatures and minimal areas of complete surfaces*, Proc. Amer. Math. Soc. **94** (1985), no. 2, 310–316. MR 784184 (86h:53047)

[Spi79] Michael Spivak, *A comprehensive introduction to differential geometry. Vol. I*, second ed., Publish or Perish Inc., Wilmington, Del., 1979. MR 532830 (82g:53003a)

[SY77] Yum Tong Siu and Shing Tung Yau, *Complete Kähler manifolds with nonpositive curvature of faster than quadratic decay*, Ann. of Math. (2) **105** (1977), no. 2, 225–264. MR 0437797 (55 #10719)

[Tka94] Vladimir G. Tkachev, *Finiteness of the number of ends of minimal submanifolds in Euclidean space*, Manuscripta Math. **82** (1994), no. 3-4, 313–330. MR 1265003 (95h:53012)

[War83] Frank W. Warner, *Foundations of differentiable manifolds and Lie groups*, Graduate Texts in Mathematics, vol. 94, Springer-Verlag, New York, 1983, Corrected reprint of the 1971 edition. MR 722297 (84k:58001)

[Whi87] Brian White, *Complete surfaces of finite total curvature*, J. Differential Geom. **26** (1987), no. 2, 315–326. MR 906393 (88m:53020)

[Whi88] ______, *Correction to: "Complete surfaces of finite total curvature" [J. Differential Geom.* **26** *(1987), no. 2, 315–326; MR0906393 (88m:53020)]*, J. Differential Geom. **28** (1988), no. 2, 359–360. MR 961520 (89j:53009)

[Yau75] Shing Tung Yau, *Isoperimetric constants and the first eigenvalue of a compact Riemannian manifold*, Ann. Sci. École Norm. Sup. (4) **8** (1975), no. 4, 487–507. MR 0397619 (53 #1478)