ON THE CHARACTERIZATION OF PARABOLICITY AND
HYPERBOLICITY OF SUBMANIFOLDS

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ABSTRACT. We give a set of sufficient and necessary conditions for parabolicity and hyperbolicity of a submanifold with controlled mean curvature in a Riemannian manifold with a pole and with sectional curvatures bounded from above or from below.

1. INTRODUCTION

A Riemannian manifold $M^n$ is said to be parabolic if it fails to admit positive Green’s functions. In this context, we say that the manifold $M$ is hyperbolic if and only if it is non-parabolic, i.e., hyperbolicity of $M$ is equivalent to the existence of a positive Green’s function defined on it.

In the paper [11], T. Lyons and D. Sullivan established the following (non-exhaustive) list of equivalent conditions for non-parabolicity, i.e. hyperbolicity (the so-called ‘Kelvin-Nevanlinna-Royden criteria’): The Riemannian manifold $M$ is hyperbolic if one (thence all) of the following equivalent conditions is satisfied: (a) $M$ admits a non-constant positive superharmonic function, (b) $M$ has positive capacity, i.e., there exists a non-empty precompact open set $D \subset M$ such that $Cap(D, M) > 0$, (c) the Brownian motion on $M$ is transient, i.e. there exists a non-empty precompact open set $D \subset M$.
such that the Brownian motion starting from a point in $D$ eventually leaves $D$ with a positive probability.

In particular, it can be proved that the capacity condition (b) implies that $M$ is parabolic if and only if it has vanishing capacity, namely, there exists a non-empty precompact open set $D \subseteq M$ such that $\text{Cap}(D, M) = 0$. See Subsection §2.4 in the Preliminaries for more details about the definition of capacity.

On the other hand, it is known that given $M^2$ a differentiable, two dimensional, simply connected and non-compact Riemannian manifold, then it is conformally diffeomorphic either to the Euclidean plane $\mathbb{R}^2$, (surface of parabolic type), or to the hyperbolic plane $\mathbb{H}^2$, (surface of hyperbolic type).

At this point, and to avoid confusion, we should remark that a two dimensional, simply connected, non-compact Riemannian manifold $M^2$ is of hyperbolic type if and only if $M^2$ is hyperbolic (non-parabolic) in our sense. This observation is due to the fact that in dimension two every conformal map preserves harmonic functions, so the potentials that we use for calculating the capacity, (which are harmonic functions), are preserved by conformal maps. Hence, the capacity itself is an invariant under conformal diffeomorphisms and therefore, for a surface $M^2$ is equivalent to be of hyperbolic type (resp. parabolic type) than to be hyperbolic/transient (resp. parabolic/recurrent).

However, if a Riemann surface is not simply connected, its type is determined by that of its universal cover and hence has nothing to do with our definition of parabolicity/hyperbolicity. This equivalence also does not survive in higher dimensions (e.g. $\mathbb{R}^3$ is hyperbolic). Finally, there are other generalizations of the notion of hyperbolicity, as Gromov’s hyperbolicity that could generate some mismatch in the usage of the term ‘hyperbolic’. Concerning this, we observe that there are hyperbolic manifolds which are not Gromov hyperbolic, as, e.g., the two-dimensional Jungle-gym (a $\mathbb{Z}^2$-covering
of a surface of genus two). To get a complete view of all these concepts, we refer to the surveys [5] and [10].

To find a geometric description for the parabolicity (or hyperbolicity) of a Riemannian manifold is a question which holds in a central position inside the function theory on Riemannian manifolds, as we can see in surveys [5] and [10]. This description can be given as a characterization, or as a sufficient or a necessary condition. The geometry involved encompasses concepts as the volume growth of the manifold or bounds on its Ricci or sectional curvature (see [1], [12], [5], [8], [9], [2], [21] or, more recently, [6]).

In 1935, L.V. Ahlfors proved in [1] that a rotationally symmetric surface $M^2$ is parabolic if and only if the integral $\int_0^{\infty} \frac{1}{\text{vol}(S_r)} dr$ is divergent, $S_r$ being the geodesic circle of radius $r$ in $M^2$. Based on this result, J. Milnor obtained in [12] a decision criterion for the parabolicity/hyperbolicity of a complete rotationally symmetric surface which involves its Gaussian curvature. In [3], P. G. Doyle showed how to extend this criterion to complete surfaces having a global geodesic polar coordinate system (namely, having a pole).

Ahlfors’ result has been generalized by several authors (see [5]) to rotationally symmetric spaces with dimension bigger than two (the so-called model spaces which will be presented in Subsection §2.2), so we have the following theorem:

**Theorem A** ([1], [5]). Let $M^n_w$ be a complete and non compact model space. Then $M^n_w$ is parabolic (resp. hyperbolic) if and only if

$$\int_{\rho}^{\infty} \frac{dr}{w^{n-1}(r)} = \infty \quad (\text{resp. } < \infty)$$

where the volume of the geodesic spheres $S_r^w$ of $M^n_w$ is given by $\text{vol}(S_r^w) = w^{n-1}(r)$.

Finally, K. Ichihara proved in [8] that a connected and complete $n$-Riemannian manifold is parabolic if its Ricci curvatures are bounded from below by the
corresponding curvatures of a model space which satisfies the Ahlfors’ integral divergence condition, and it is hyperbolic provided its sectional curvatures are bounded from above by the corresponding curvatures of a model space which satisfies the Ahlfors’ integral convergence condition.

1.1. **A first glimpse of the main results.** In this paper it is considered a submanifold $P^m$ properly immersed in an ambient manifold $N^n$ which has at least one pole and its radial sectional curvatures (namely, the sectional curvatures of the planes containing the radial directions from the pole) bounded from above or from below.

Then, continuing the programme started with papers [15], [16] and [17], we are going to establish a set of sufficient conditions for parabolicity and hyperbolicity of submanifolds (Theorems 3.4 and 3.6). These results partially encompass the results in [16] and [17], and the techniques used to obtain them are based, as in those papers, on the Hessian and Laplacian comparison theory of restricted distance function, which involves bounds on the mean curvature of the submanifold.

The way to prove Theorem 3.4 and Theorem 3.6 (which are the main results of this paper) consists in the application of the above Kelvin-Nevanlinna-Royden Criteria (see also [5, Theorem 5.1]) proving the hyperbolicity of the submanifold by showing that $P$ has *positive capacity*, i.e. by showing the existence of a non-empty precompact set in the submanifold with positive capacity, and proving the parabolicity of $P$ by showing that $P$ has *vanishing capacity*, i.e., by showing the existence of a non-empty precompact set in $P$ with zero capacity. This method, which encompasses the use of the distance function from the pole, restricted to the submanifold, is inspired in the *Rayleigh’s short-cut method* from the classical theory of electricity, used by J. Milnor in [12] and by P. G. Doyle in [3].
As a consequence of these results, and using the logical interplay among them and the definitions of hyperbolicity and parabolicity, we have obtained two corollaries (Corollaries 3.11 and 3.9) with necessary conditions for these properties. All these results together attempt to approach a geometric characterization of parabolicity and hyperbolicity for submanifolds in an ambient manifold with bounded (above or below) sectional curvatures, in the style of Theorem A.

This geometric characterization could, in the first instance, shed light on the following facts concerning the parabolicity/hyperbolicity of the minimal surfaces in the Euclidean three space $\mathbb{R}^3$. It was proved in [15] that minimal submanifolds of Cartan-Hadamard manifolds are hyperbolic. It must be remarked that the case of minimal surfaces in $\mathbb{R}^3$ (and in $\mathbb{R}^n$ in general) is excluded from this result. For example, while the catenoid is parabolic, the doubly periodic Scherk’s surface or the triply periodic Schwarz $\mathcal{P}$-surface are hyperbolic. However, the minimal surfaces of the hyperbolic 3-space are hyperbolic.

In order to explain this particular behaviour, some control on the ‘radiality’ of the submanifold was introduced in the paper [17], (see too [7]). This ‘radiality’ means the following: Assuming for the sole purpose of this explanation (the proof of our results is independent of the position of the pole) that the pole $o$ of the ambient manifold lies in submanifold $P$, when $P$ is totally geodesic, then $\nabla^N r = \nabla^P r$ in all points, and, hence, $\|\nabla^P r\| = 1$. On the other hand, and given the starting point $o \in P$ from which we are measuring the distance $r$, we know that $\nabla^N r(o) = \nabla^P r(o)$, so $\|\nabla^P r(o)\| = 1$. Therefore, the difference $1 - \|\nabla^P r\|$ quantifies the radial detour of the submanifold with respect to the ambient manifold as seen from the pole $o$. To control this detour locally, we apply the following
Definition 1.1. We say that the submanifold $P$ satisfies a radial tangency condition at $o \in N$ when we have a smooth positive function,

$$g : P \mapsto \mathbb{R}_+,$$

so that

$$T(x) = \|\nabla^P r(x)\| \geq g(r(x)) > 0 \text{ for all } x \in P.$$

We also need the precise definition of the radially weighted component of mean curvature, which plays an important rôle in the Laplacian inequalities established in section §3.2:

Definition 1.2. The $o$-radial mean convexity $C(x)$ of $P$ in $N$ is defined in terms of the inner product of $H_P$ with the $N$-gradient of the distance function $r(x)$ as follows:

$$C(x) = -\langle \nabla^N r(x), H_P(x) \rangle, \quad x \in P,$$

where $H_P(x)$ denotes the mean curvature vector of $P$ in $N$.

Note that the $o$-radial mean convexity of a minimal submanifold $P$ is $C(x) = 0 \forall x \in P$.

Related with the radial convexity form the pole, we have the following

Definition 1.3. (1) The submanifold $P$ is called radially $0$-convex if and only if $C(x) \geq 0 \forall x \in P$. This condition is satisfied by convex hypersurfaces of real space forms $\mathbb{K}^m(b)$ of constant curvature $b$ (see [19]), as well as by all minimal submanifolds.

(2) The submanifold $P$ is called radially minimal if and only if $C(x) = 0 \forall x \in P$. This condition is satisfied by all minimal submanifolds.

With these initial definitions we can now state the following first instances of our parabolicity and hyperbolicity criteria, trying to facilitate intuition
concerning our main results by considering some of their hypothesis and consequences restricted to radially 0-convex and radially minimal submanifolds $P^m$ properly immersed in the simply connected real space forms with constant curvature $b \leq 0$, which we shall denote as $\mathbb{K}^n(b)$. The results presented here are but shadows of the general results, Theorem 3.4 and Theorem 3.6 stated and proved in sections §4 and §5. In our general statements, we give parabolicity and hyperbolicity criteria for properly immersed submanifolds with o-radial mean convexity bounded from below or from above by a balanced radial function $h(r)$. The corresponding ambient spaces are Riemannian manifolds with sectional curvatures bounded from above or from below by the radial sectional curvatures of a model space (see Subsection §2.2).

**Theorem 1.4.** Let $P^m$ be a complete, properly immersed and radially 0-convex submanifold in $\mathbb{K}^n(b)$, $(b \leq 0)$. Assume that the submanifold $P$ satisfies a radial tangency condition at $o \in N$, (namely, there exists smooth $g : P \mapsto \mathbb{R}_+$, so that $\|\nabla^P r(x)\| \geq g(r(x)) > 0$ for all $x \in P$), and that

\[ \int_{\rho}^{\infty} \Lambda_g(t) \, dt = \infty, \]

where

\[ \Lambda_g(r) = w_b(r) \exp \left( -m \int_{\rho}^{r} \frac{\eta_{wb}(t)}{g^2(t)} \, dt \right). \]

being $\eta_{wb}(r)$ the constant mean curvature of the distance spheres in $\mathbb{K}^n(b)$. Then $P^m$ is parabolic.

**Remark 1.5.** Theorem 1.4 will follows directly from Theorem 3.4, using the radial constant function $h(r) = 0 \forall r$ to bound the o-radial mean convexity of $P$. Note that this function is balanced from above with respect to the warping function $w_b(r)$ associated to the ambient manifold $\mathbb{K}^n(b)$. 
Theorem 1.6. Let $P^m$ be a complete, properly immersed and radially minimal submanifold in $\mathbb{K}^n(b)$, $(b \leq 0)$. Then $P^m$ is hyperbolic if either $(b < 0$ and $m \geq 2)$ or $(b = 0$ and $m \geq 3)$.

Remark 1.7. Theorem 1.6 follows from statement (B) of Theorem 3.6, using as in Theorem 1.4 the balanced from above function $h(r) = 0 \forall r$. Note that when we consider the function $\Lambda(r) = w_b(r) \exp \left( -m \int_{\rho}^{r} \eta w_b(t) dt \right)$ we conclude, if $b < 0$, that $\int_{\rho}^{\infty} \Lambda(t) dt < \infty \forall m \geq 2$ and if $b = 0$ then $\int_{\rho}^{\infty} \Lambda(t) dt < \infty \forall m \geq 3$. In this sense, Theorem 1.6 is a particular instance of Theorem 2.1 in [15].

As a consequence of Theorem 1.4, we have the following result when we consider surfaces in 3-dimensional Euclidean space. This corollary is, in its turn, a particular case of Corollary 3.9.

Corollary 1.8. Let $P^2$ be a complete, properly immersed and radially 0-convex surface of $\mathbb{R}^3$. If $P^2$ is hyperbolic, then

(i) either there does not exist a smooth positive function $g : P \mapsto \mathbb{R}^+$, so that $\|\nabla^P r(x)\| \geq g(r(x)) > 0$ for all $x \in P$

(ii) or, in case $P$ satisfies a radial tangency condition at $o \in P$ with smooth positive function $g : P \mapsto \mathbb{R}^+$, then $\int_{\rho}^{\infty} re^{-\int_{\rho}^{r} \frac{2}{\eta w_b(t)} dt} dr < \infty$.

Remark 1.9. In Corollary 2.2 of [17] it was proved that a two-dimensional surface $P^2$ in the Euclidean space with the radial component of its mean curvature $H_P$ bounded from below by 0 is parabolic if the lower bound for its radial tangency $T(x)$ is a radial function $g(r)$ which is close to 1 at infinity.

By contrast, (as was pointed out there), the Scherk’s doubly periodic minimal surface is a hyperbolic surface in $\mathbb{R}^3$, such that its radial tangency
(from any fixed point $o$ in the $(x, y)$-plane) is “mostly” close to 1 at infinity, except for the points in the $(x, y)$-plane itself, where the tangency function is close to 0. Corollary 1.8 intended to provide a partial explanation of the particular behavior of the Scherk’s surface.

On the other hand, in the same article [17] an example is shown of how the catenoid, a minimal and parabolic surface, satisfies a radial tangency condition at the origin $\bar{0} \in \mathbb{R}^3$.

**Example 1.10.** As an example of a surface where it is easy to see that this result holds, we have Schwarz P-surface $\mathcal{P} \subseteq \mathbb{R}^3$. This is a triply periodic minimal surface which is hyperbolic. Its unit cell, constructed by solving the Plateau problem for a square with corners at the vertices of a regular octahedron, can be viewed roughly as a sphere $S^2$ from which six spherical caps whose centroids are antipodal in pairs have been removed. In the web page [22], we can see an image of this unit cell, with the surface-generating straight boundary lines.

We are going to see that assertion (i) of Corollary 1.8 holds for this surface. To do that, it must be remarked first that all the points in the ambient space $\mathbb{R}^3$ are poles. Then, if we consider the center of our extrinsic balls as the center of one of these spheres-unit cells, there exist at least eight points on the surface of this unit cell (the points where three of the generating straight lines intersect) where $\nabla_{\mathbb{R}^3} r$ is orthogonal to $\mathcal{P}$. Hence, assertion (i) of Corollary 1.8 is satisfied.

1.2. **Outline of the paper.** We shall present the basic definitions and results which are in the foundations of our developments in Section 2. Section 3 is devoted to the statement of main theorems and its corollaries. Proofs of main theorems 3.4 and 3.6 are presented in Sections 4 and 5.
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2. **Preliminaries**

We assume throughout the paper that $P^m$ is a non-compact, properly immersed, Riemannian submanifold of a complete Riemannian manifold $N^n$. Furthermore, we assume that $N^n$ possesses at least one pole. Let it be remembered that a pole is a point $o$ such that the exponential map $\exp_o : T_oN^n \to N^n$ is a diffeomorphism. For every $x \in N^n \setminus \{o\}$ we define $r(x) = \text{dist}_N(o, x)$, and this distance is realized by the length of a unique geodesic from $o$ to $x$, which is the **radial geodesic from $o$**. We also denote by $r$ the restriction $r|_P : P \to \mathbb{R}_+ \cup \{0\}$. This restriction is called the **extrinsic distance function** from $o$ in $P^m$. The gradients of $r$ in $N$ and $P$ are denoted by $\nabla^N r$ and $\nabla^P r$, respectively. Let us remark that $\nabla^P r(x)$ is just the tangential component in $P$ of $\nabla^N r(x)$ for all $x \in P$. Then we have the following basic relation:

\( (2.1) \quad \nabla^N r = \nabla^P r + (\nabla^N r)^\perp, \)

where $(\nabla^N r)^\perp(x)$ is perpendicular to $T_x P$ for all $x \in P$.

2.1. **Curvature restrictions and extrinsic balls.**

**Definition 2.1.** Let $o$ be a point in a Riemannian manifold $M$ and let $x \in M \setminus \{o\}$. The sectional curvature $K_M(\sigma_x)$ of the two-plane $\sigma_x \in T_x M$ is then called an **$o$-radial sectional curvature** of $M$ at $x$ if $\sigma_x$ contains the tangent vector to a minimal geodesic from $o$ to $x$. We denote these curvatures by $K_{o,M}(\sigma_x)$. 
**Definition 2.2.** (1) The submanifold $P$ is called \textit{radially 0-convex} if and only if $C(x) \geq 0 \forall x \in P$. This condition is satisfied by \textit{convex} hypersurfaces of real space forms $\mathbb{K}^m(b)$ of constant curvature $b$ (see [19]), as well as by all \textit{minimal} submanifolds.

(2) The submanifold $P$ is called \textit{radially minimal} if and only if $C(x) = 0 \forall x \in P$. This condition is satisfied by all \textit{minimal} submanifolds.

**Definition 2.3.** Given a connected and complete $m$-dimensional submanifold $P^m$ in a complete Riemannian manifold $N^n$ with a pole $o$, we denote the \textit{extrinsic metric balls} of (sufficiently large) radius $R$ and center $o$ by $D_R(o)$. They are defined as any connected component of the intersection

$$B_R(o) \cap P = \{x \in P : r(x) < R\},$$

where $B_R(o)$ denotes the open geodesic ball of radius $R$ centered at the pole $o$ in $N^n$. Using these extrinsic balls we define the $o$-centered extrinsic annuli

$$A_{\rho,R}(o) = D_R(o) \setminus \bar{D}_\rho(o)$$

in $P^m$ for $\rho < R$, where $D_R(o)$ is the component of $B_R(o) \cap P$ containing $D_\rho(o)$.

**Remark 2.4.** We want to point out that the extrinsic domains $D_R(o)$ are precompact sets (because the submanifold $P$ is properly immersed) with smooth boundary $\partial D_R(o)$. The assumption on the smoothness of $\partial D_R(o)$ makes no restriction. Indeed, the distance function $r$ is smooth in $N^n \setminus \{o\}$ since $N^n$ is assumed to possess a pole $o \in N^n$. Hence the restriction $r|_P$ is smooth in $P$ and consequently the radii $R$ that produce smooth boundaries $\partial D_R(o)$ are dense in $\mathbb{R}$ by Sard’s theorem and the Regular Level Set Theorem.
Upper and lower bounds on $C(x)$ and $T(x)$ together with a suitable control on the $o$-radial sectional curvatures of the ambient space will eventually control the Laplacian of restricted radial functions on $P$.

2.2. Warped products and model spaces. Warped products are generalized manifolds of revolution, see e.g. [18]. Let $(B^k, g_B)$ and $(F^l, g_F)$ denote two Riemannian manifolds and let $w: B \to \mathbb{R}_+$ be a positive real function on $B$. We assume throughout that $w$ is at least $C^2$. We consider the product manifold $M^{k+l} = B \times F$ and denote the projections onto the factors by $\pi: M \to B$ and $\sigma: M \to F$, respectively. The metric $g$ on $M$ is then defined by the following $w$-modified (warped) product metric

$$g = \pi^*(g_B) + (w \circ \pi)^2 \sigma^*(g_F).$$

**Definition 2.5.** The Riemannian manifold $(M, g) = (B^k \times F^l, g)$ is called a warped product with warping function $w$, base manifold $B$ and fiber $F$. We write as follows: $M^m_w = B^k \times_w F^l$.

**Definition 2.6.** (See [5], [4]). A $w$-model $M^m_w$ is a smooth warped product with base $B^1 = [0, \Lambda] \subset \mathbb{R}$ (where $0 < \Lambda \leq \infty$), fiber $F^{m-1} = S^{m-1}_1$ (i.e. the unit $(m-1)$-sphere with standard metric), and warping function $w: [0, \Lambda] \to \mathbb{R}_+ \cup \{0\}$, with $w(0) = 0$, $w'(0) = 1$, and $w(r) > 0$ for all $r > 0$. The point $o_w = \pi^{-1}(0)$, where $\pi$ denotes the projection onto $B^1$, is called the center point of the model space. If $\Lambda = \infty$, then $o_w$ is a pole of $M^m_w$.

**Proposition 2.7.** The simply connected space forms $K^m(b)$ of constant curvature $b$ are $w$-models with warping functions

$$w(r) = w_b(r) = \begin{cases} \frac{1}{\sqrt{b}} \sin(\sqrt{b} r) & \text{if } b > 0 \\ r & \text{if } b = 0 \\ \frac{1}{\sqrt{-b}} \sinh(\sqrt{-b} r) & \text{if } b < 0. \end{cases}$$
Note that for $b > 0$ the function $w_b(r)$ admits a smooth extension to $r = \pi / \sqrt{b}$.

**Proposition 2.8** (See [18], [4] and [5]). Let $M^m_w$ be a $w-$model with warping function $w(r)$ and center $o_w$. The distance sphere of radius $r$ and center $o_w$ in $M^m_w$ is the fiber $\pi^{-1}(r)$. This distance sphere has the constant mean curvature $\eta_w(r) = \frac{w'(r)}{w(r)}$. On the other hand, the $o_w$-radial sectional curvatures of $M^m_w$ at every $x \in \pi^{-1}(r)$ (for $r > 0$) are all identical and determined by

$$K_{o_w,M^m_w}(\sigma_x) = -\frac{w''(r)}{w(r)}.$$

2.3. Hessian and Laplacian comparison analysis. The 2nd order analysis of the restricted distance function $r_p$, defined on manifolds with a pole is first and foremost governed by the Hessian comparison Theorem A in [4]:

**Theorem 2.9** (See [4], Theorem A). Let $N = N^n$ be a manifold with a pole $o$, let $M = M^m_w$ denote a $w-$model with center $o_w$, and $m \leq n$. Suppose that every $o$-radial sectional curvature at $x \in N \setminus \{o\}$ is bounded from above by the $o_w$-radial sectional curvatures in $M^m_w$ as follows:

$$K_{o,N}(\sigma_x) \geq (\leq) -\frac{w''(r)}{w(r)}$$

for every radial two-plane $\sigma_x \in T_xN$ at distance $r = r(x) = \text{dist}_N(o,x)$ from $o$ in $N$. Then the Hessian of the distance function in $N$ satisfies

$$\text{Hess}^N(r(x))(X,X) \leq (\geq) \text{Hess}^M(r(y))(Y,Y)$$

(2.2)  

$$= \eta_w(r) \left(1 - \langle \nabla^M r(y), Y \rangle^2_M \right)$$

$$= \eta_w(r) \left(1 - \langle \nabla^N r(x), X \rangle^2_N \right)$$

for every unit vector $X$ in $T_xN$ and for every unit vector $Y$ in $T_yM$ with $r(y) = r(x) = r$ and $\langle \nabla^M r(y), Y \rangle_M = \langle \nabla^N r(x), X \rangle_N$.

**Remark 2.10.** In [4, Theorem A, p. 19], the Hessian of $r_M$ is less than or equal to the Hessian of $r_N$ provided that the radial curvatures of $N$
are bounded from above by the radial curvatures of $M$ and provided that $\dim M \geq \dim N$. This latter dimension condition is not satisfied in our setting. However, since $(M^m, g)$ is a $w-$model space it has an $n-$dimensional $w-$model space companion with the same radial curvatures and the same Hessian of radial functions as $(M^m, g)$. In effect, therefore, applying [4, Theorem A, p. 19] to the high-dimensional comparison space gives the low-dimensional comparison inequality as stated.

In other words, $\text{Hess}^{M_w}(r(y))(Y, Y)$ does not depend on the dimension $m$, as we can easily see by computing it directly (see [20]), so the hypothesis on the dimension can be overlooked in the comparison among the Hessians.

As a consequence of this result, we have the following Laplacian inequalities:

**Proposition 2.11.** Let $N^n$ be a manifold with a pole $p$, let $M^m_w$ denote a $w-$model with center $p_w$, and let $P^m$ be a properly immersed submanifold in $N^n$.

(i) Suppose that every $o-$radial sectional curvature at $x \in N \setminus \{o\}$ is bounded from below by the $o_w$-radial sectional curvatures in $M^m_w$ as follows:

$$K(\sigma(x)) = K_{o,N}(\sigma_x) \geq -\frac{w''(r)}{w(r)},$$

for every radial two-plane $\sigma_x \in T_x N$ at distance $r = r(x) = \text{dist}_N(o, x)$ from $o$ in $N$. Then we have for every smooth function $f(r)$ with $f'(r) \leq 0$ for all $r$ (respectively $f'(r) \geq 0$ for all $r$):

$$\Delta^P(f \circ r) \geq (\leq) (f''(r) - f'(r)\eta_w(r)) \|\nabla^P r\|^2$$

$$+ mf'(r) (\eta_w(r) + \langle \nabla^N r, H_P \rangle),$$

where $H_P$ denotes the mean curvature vector of $P$ in $N$. 

(ii) Suppose that every $o$-radial sectional curvature at $x \in N \setminus \{o\}$ is bounded from above by the $o_w$-radial sectional curvatures in $M^m_w$ as follows:

$$\mathcal{K}(\sigma(x)) = K_{o,N}(\sigma_x) \leq -\frac{w''(r)}{w(r)}$$

for every radial two-plane $\sigma_x \in T_xN$ at distance $r = r(x) = \text{dist}_N(o,x)$ from $p$ in $N$. Then we have for every smooth function $f(r)$ with $f'(r) \leq 0$ for all $r$ (respectively $f'(r) \geq 0$ for all $r$):

$$\Delta^P(f \circ r) \leq (\geq) \left( f''(r) - f'(r)\eta_w(r) \right) \|\nabla^P r\|^2$$

$$+ m f'(r) \left( \eta_w(r) + \langle \nabla^N r, H_P \rangle \right).$$

2.4. Capacities of extrinsic annular domains. The proof of theorems 3.4 and 3.6 is based on the existence of (lower and upper) bounds for the capacity of some precompact subset in the submanifold $P^m$. This precompact subset is an extrinsic ball $D_\rho(o) \subseteq S$.

In general, the capacity of a compact domain $K$ in a precompact open set $\Omega$ of a Riemannian manifold $M$ can be expressed as the following integral along the boundary of the compact set $K$ (see e.g. [5]):

$$\text{Cap}(K, \Omega) = \int_{\Omega \setminus K} \|\nabla^M v\|^2 dV,$$

where the function $v$ is the solution of the Dirichlet problem in $\Omega - K$

$$\begin{cases} 
\Delta^M v = 0 & \text{on } \Omega \setminus K \\
v = 0 & \text{on } \partial K \\
v = 1 & \text{on } \partial \Omega.
\end{cases}$$

The capacity of $K$ in the whole manifold $M$ is given by the following limit, given any exhaustion sequence of precompact open subsets $\{\Sigma_n\}_{n \in \mathbb{N}}$ covering all of $M$ such that $\Sigma_0 = K$ and $\Sigma_n \subseteq \Sigma_{n+1}$, (see [5]):

$$\text{Cap}(K, M) = \lim_{n \to \infty} \text{Cap}(K, \Sigma_n).$$
Using the divergence theorem, it is easy to see, ([5]), that integral (2.7) becomes

\[(2.10) \quad \text{Cap}(K, \Omega) = \int_{\partial K} \langle \nabla^M v, \nu \rangle dA ,\]

where \(\nu\) is the unit normal vector field on \(\partial K\) which points into the domain \(\Omega \setminus K\).

In our setting, we have a compact set in the submanifold \(P^m, \bar{D}_\rho(o)\), and an exhaustion of \(P\) given by the extrinsic balls \(\{D_R(o)\}_{R>0}\) which contains \(\bar{D}_\rho(o)\). The computation of the capacity of these extrinsic annular domains \(A_{\rho,R} = D_R(o) \setminus \bar{D}_\rho(o)\) is given by the following considerations, applying equation (2.10):

\[(2.11) \quad \text{Cap}(A_{\rho,R}) = \int_{\partial D_\rho} \langle \nabla^P v, n_{\partial D_\rho} \rangle_{\partial D_\rho} d\mu ,\]

where \(v(x)\) is the Laplace potential function for the extrinsic annulus \(A_{\rho,R} = D_R \setminus D_\rho\), setting \(v_{\partial D_\rho} = 0\) and \(v_{\partial D_R} = 1\) and \(n_{\partial D_\rho}\) denotes the unit normal vector field along \(\partial D_\rho\) pointing into the domain \(A_{\rho,R}\).

The function \(v\) must be nonnegative in the annular domain \(A_{\rho,R}\). Otherwise \(v\) would have an intrinsic (negative) minimum in \(A_{\rho,R}\), and since \(v\) is harmonic this is ruled out by the minimum principle.

Now, since \(v\) is nonnegative and \(v = 0\) at the inner boundary, then the inwards directed gradient \(\langle \nabla^P v, n_{\partial D_\rho} \rangle_{\partial D_\rho}\) is also nonnegative. Since \(\partial D_\rho\) is a level hypersurface (of value \(v = 0\)) for \(v\) in \(P\), we have that \(n_{\partial D_\rho}\) is proportional to \(\nabla^P v\). It therefore follows that

\[(2.12) \quad \langle \nabla^P v, n_{\partial D_\rho} \rangle_{\partial D_\rho} = \| \nabla^P v(x) \| .\]

Therefore we have

\[(2.13) \quad \text{Cap}(A_{\rho,R}) = \int_{\partial D_\rho} \| \nabla^P v(x) \| \ d\nu .\]
3. Main results

We are going to provide some previous definitions in order to formulate our main hyperbolicity and parabolicity results. The proofs are developed through the following sections.

**Definition 3.1.** Let $N^n$ be a complete manifold with pole $o \in N$, and let $P^m$ be a properly immersed submanifold in $N$. Given a function $h : P \to \mathbb{R}$ which only depends on the extrinsic distance $r$ in $P$, $h(r(x))$ for all $x \in P$, we claim that the function $h(r)$ is balanced from above with respect to the warping function $w(r)$ of a model space $M^m_w$ if

$$M(r) = m(\eta_w(r) - h(r)) \geq 0 \quad \forall r$$

and that the function $h(r)$ is balanced from below with respect to the warping function $w(r)$ if

$$M(r) = m(\eta_w(r) - h(r)) \leq 0 \quad \forall r$$

**Definition 3.2.** Let $N^n$ be a complete manifold with pole $o \in N$, and let $P^m$ be a properly immersed submanifold in $N$. Let us also consider a model space $M^m_w$.

(i) Define $\Lambda(r)$ as the function

$$\Lambda(r) = w(r) \exp \left( - \int_{\rho}^{r} M(t) \, dt \right).$$

(ii) Assume moreover that $P$ satisfies a radial tangency condition at $o \in N$. We denote as $\Lambda_g(r)$ the function

$$\Lambda_g(r) = w(r) \exp \left( - \int_{\rho}^{r} \frac{M(t)}{g^2(t)} \, dt \right).$$

**Remark 3.3.** In the following parabolicity and hyperbolicity criteria a fundamental rôle is played by the convergence/divergence of the infinite integrals $\int_{\rho}^{\infty} \Lambda_g(t) \, dt$ and $\int_{\rho}^{\infty} \Lambda(t) \, dt$. Therefore, we should remark that, if
$\mathcal{M}(r) \geq 0$ for all $r > 0$, then $\int_{\rho}^{\infty} \Lambda_\varrho(t) \, dt \leq \int_{\rho}^{\infty} \Lambda(t) \, dt$ and, on the other hand, if $\mathcal{M}(r) \leq 0$ for all $r > 0$, then $\int_{\rho}^{\infty} \Lambda_\varrho(t) \, dt \geq \int_{\rho}^{\infty} \Lambda(t) \, dt$. Let it be remembered at this point that the $o$-radial mean convexity $C(x)$ of $P$ in $N$ is defined in definition 1.2 in the Introduction.

**Theorem 3.4** (Parabolicity). Let $N^n$ be a complete manifold with pole $o$, and suppose that

\[(3.3)\quad K_{o,N}(\sigma_x) \geq \frac{-w''(r)}{w(r)}\]

for all $x$ with $r = r(x) \in [0, \infty)$.

Let $P^m$ be a complete and properly immersed submanifold with $o$-radial mean convexity $C(x)$ bounded from below by the radial function $h(r(x))$:

\[(3.4)\quad C(x) \geq h(r(x)) \quad \text{for all} \quad x \in P^m \quad \text{with} \quad r(x) \in [0, \infty)\]

Then:

(A) Assume that the submanifold $P$ satisfies a radial tangency condition at $o \in N$ (namely, there exists smooth $g : P \mapsto \mathbb{R}_+$, so that $\|\nabla^P r(x)\| \geq g(r(x)) > 0$ for all $x \in P$) and that the function $h(r)$ is balanced from above with respect to the warping function $w(r)$, $(\mathcal{M}(r) \geq 0 \forall r)$.

Suppose that

\[(3.5)\quad \int_{\rho}^{\infty} \Lambda_\varrho(t) \, dt = \infty\]

Then $P^m$ is parabolic.

(B) Assume that the function $h(r)$ is balanced from below with respect to the warping function $w(r)$ $(\mathcal{M}(r) \leq 0 \forall r)$ and suppose that

\[(3.6)\quad \int_{\rho}^{\infty} \Lambda(t) \, dt = \infty\]

Then $P^m$ is parabolic.
Remark 3.5. Theorem 3.4 (A) has been stated and proved in [17], (see Theorem 9.2), under a more restrictive balance condition.

**Theorem 3.6 (Hyperbolicity).** Let $N^n$ be a complete manifold with pole $o$, and suppose that

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r)}{w(r)}$$

for all $x$ with $r = r(x) \in [0, \infty)$. Let $P^m$ be a complete and properly immersed submanifold with o-radial mean convexity $C(x)$ bounded from above by the radial function $h(r(x))$:

$$C(x) \leq h(r(x)) \text{ for all } x \in P^m \text{ with } r(x) \in [0, \infty).$$

Then

(A) Assume that the submanifold $P$ satisfies a radial tangency condition at $o \in P$, and that the function $h(r)$ is balanced from below with respect to the warping function $w(r)$, $(\mathcal{M}(r) \leq 0 \forall r)$.

Suppose finally that

$$\int_\rho^\infty \Lambda_g(t) \, dt < \infty.$$ 

Then $P^m$ is hyperbolic.

(B) Assume that the function $h(r)$ is balanced from above with respect to the warping function $w(r)$, $(\mathcal{M}(r) \geq 0 \forall r)$, and suppose that

$$\int_\rho^\infty \Lambda(t) \, dt < \infty.$$ 

Then $P^m$ is hyperbolic.

**Remark 3.7.** Theorem 3.6 (B) has been stated and proved in [16]. If we follow the notation in [16], we have

$$G(r) = \exp(\int_\rho^r h(t) \, dt)$$
and it is straightforward to check that $\int_\rho^\infty \frac{g^m(r)}{w^{m-1}(r)} \, dr < \infty$ iff $\int_\rho^\infty \Lambda(t) \, dt < \infty$.

**Remark 3.8.** We have the following examples of a direct application of theorems 3.4 and 3.6. Concerning Theorem 3.4, we can see that the cones and the paraboloids (both convex hypersurfaces in $\mathbb{R}^3$) are parabolic (see [17]). Concerning Theorem 3.6, we have that surfaces $P^2$ in $\mathbb{H}^3(b)$ with constant mean curvature $H_P \leq \frac{1}{2}\sqrt{-b}$ are hyperbolic (see too Corollary B in [16]).

### 3.1. Corollaries

Finally, as corollaries of Theorem 3.4 and Theorem 3.6, we have the following results.

**Corollary 3.9.** Let $N^n$ be a complete manifold with pole $o$, and let $P^m$ be a properly immersed submanifold in $N$, both satisfying inequalities (3.3) and (3.4) in Theorem 3.4.

(A) Let us suppose that $\mathcal{M}(r) \geq 0 \forall r$.

If $P$ is hyperbolic, then

(A.1) either there does not exist a smooth positive function $g : P \mapsto \mathbb{R}_+$, so that $\|\nabla^S r(x)\| \geq g(r(x)) > 0$ for all $x \in P$

(A.2) or, if $P$ satisfies a radial tangency condition at $o \in P$ with smooth positive function $g : P \mapsto \mathbb{R}_+$, then $\int_\rho^\infty \Lambda_g(r) \, dr < \infty$.

(B) Let us suppose that $\mathcal{M}(r) \leq 0 \forall r$.

If $P$ is hyperbolic, then $\int_\rho^\infty \Lambda(r) \, dr < \infty$.

**Proof.** (A) The ambient manifold $N$ and the submanifold $P$ satisfy hypotheses (3.3) and (3.4) in Theorem 3.4. Hence, if $P$ is not parabolic, then we have the negation of both sets of assumptions in assertions (A) and (B) in
Theorem 3.4. In particular, assertion (A) doesn’t hold. As $M(r) \geq 0 \forall r$, then either there is not any smooth positive function

$$g : P \mapsto \mathbb{R}_+,$$

so that $\|\nabla^S r(x)\| \geq g(r(x)) > 0$ for all $x \in P$ or, if this bounding function for the tangency exists, then $\int_0^\infty \Lambda g(r)dr < \infty$.

(B) In this case, $M(r) \leq 0 \forall r$, and, as assertion (B) in Theorem 3.4 doesn’t hold, we conclude that $\int_0^\infty \Lambda(r)dr < \infty$. \qed

Remark 3.10. Corollary 1.8 in the Introduction follows from Corollary 3.9, if we consider that the ambient manifold $N$ is the Euclidean 3-space $\mathbb{R}^3$ (which implies considering $w(r) = r$ as a warping function) and we take into account that, by hypothesis, $h(r(x)) = 0$ for all $x \in P$, and hence $C(x) = \eta_w(r(x)) - h(r(x)) = \frac{1}{r} > 0$ for all $x \in P$.

Corollary 3.11. Let $N^n$ be a complete manifold with pole $o$, and let $P^m$ be a properly immersed submanifold in $N$, both satisfying inequalities (3.7) and (3.8) in Theorem 3.6.

(A) Let us suppose that $M(r) \geq 0 \forall r$.

If $P$ is parabolic, then

$$\int_0^\infty \Lambda(t) dt = \infty.\quad (3.11)$$

(B) Let us suppose that $M(r) \leq 0 \forall r$.

If $P$ is parabolic, then

(B.1) either there does not exist a smooth positive function

$$g : P \mapsto \mathbb{R}_+,$$

so that $\|\nabla^S r(x)\| \geq g(r(x)) > 0$ for all $x \in P$.
(B.2) or, if $P$ satisfies a radial tangency condition at $o \in P$ with smooth positive function $g : P \mapsto \mathbb{R}_+$, then $\int_0^\infty \Lambda_g(r) dr = \infty$.

**Proof.** (A) The ambient manifold $N$ and the submanifold $P$ satisfy hypotheses (3.7) and (3.8) in Theorem 3.6. Hence, if $P$ is not hyperbolic, then we have the negation of both sets of assumptions in assertions (A) and (B) in Theorem 3.6. In particular, one of the two assumptions in assertion (B) does not hold. As $M(r) \geq 0 \forall r$, we have

\[
\int_0^\infty \Lambda(t) dt = \infty.
\]

(B) In this case, $M(r) \leq 0 \forall r$, and concluding the negation of assertion (A) in Theorem 3.6 as above, we have two possibilities: the first is that the submanifold $P$ does not satisfies a radial tangency condition at $o \in P$. Hence does not exist a smooth positive function $g : P \mapsto \mathbb{R}_+$, such that $\|\nabla^P r(x)\| \geq g(r(x)) > 0$ for all $x \in P$.

If, on the contrary, $P$ satisfies a radial tangency condition at $o \in P$ with smooth positive function $g : P \mapsto \mathbb{R}_+$, then $\int_0^\infty \Lambda_g(r) dr = \infty$. □

**Corollary 3.12.** Let $M^n_w$ be a model space, and let $P^m$ be a properly immersed submanifold of $M^n_w$. Assume that the o-radial mean convexity of $P$ in $M^n_w$ is equal to the radial function $\eta_w(r(x))$, namely

\[
C(x) = \eta_w(r(x)) \text{ for all } x \in P^m \text{ with } r(x) \in [0, \infty).
\]

Then, $P^m$ is parabolic if and only if $\int_0^\infty w(r) dr = \infty$.

4. **Proof of assertion (A) in Theorem 3.4 and Theorem 3.6**

As the submanifold $P$ satisfies a radial tangency condition, given the function $g : P \mapsto \mathbb{R}_+$, we define a second order differential operator on functions of one real variable as follows:
\[(4.1) \quad L_g \psi(r) = \psi''(r) + \psi'(r) \left( \frac{M(r)}{g^2(r)} - \eta_w(r) \right).\]

Consider the smooth solution \(\psi_{\rho,R}(r)\) of the following Dirichlet-Poisson problem associated to \(L_g\):

\[
\begin{aligned}
L_g \psi &= 0 \quad \text{on } [\rho, R] \\
\psi(\rho) &= 0 \\
\psi(R) &= 1
\end{aligned}
\]

The explicit solution to the Dirichlet problem (4.2) is given in the following Proposition which is straightforward.

**Proposition 4.1.** The solution to the Dirichlet problem (4.2) only depends on \(r\) and is given explicitly - via the function \(\Lambda_g(r)\) introduced in Definition 3.2 (ii), by:

\[(4.3) \quad \psi_{\rho,R}(r) = \frac{\int_{\rho}^{R} \Lambda_g(t) \, dt}{\int_{\rho}^{R} \Lambda_g(t) \, dt}.\]

The corresponding 'drifted' capacity is

\[
\text{Cap}_{L_g}(A^w_{\rho,R}) = \int_{\partial D^w_{\rho}} \langle \nabla M \psi_{\rho,R}, \nu \rangle \, dA
\]

\[
= \text{Vol}(\partial D^w_{\rho}) \Lambda_g(\rho) \left( \int_{\rho}^{R} \Lambda_g(t) \, dt \right)^{-1}.
\]

At this point the proof of these two results splits, in the following way:

**Assertion (A) in Theorem 3.4.**

Concerning the proof of assertion (A) in Theorem 3.4 it is easy to see, using equation (4.3) and the balance condition (3.1), that the solution \(\psi_{\rho,R}\) of the
problem (4.2) satisfies:

\[
\begin{align*}
\psi'_{\rho,R}(r) & \geq 0 \\
\psi''_{\rho,R}(r) - \psi'_{\rho,R}(r)\eta_w(r) & = -\psi'_{\rho,R}(r)\frac{M(r)}{g^2(r)} \leq 0
\end{align*}
\] (4.5)

Now we transplant the model space solutions \( \psi_{\rho,R}(r) \) of equation (4.2) into the extrinsic annulus \( A_{\rho,R} = D_R(o) \setminus \bar{D}_\rho(o) \) in \( P \) by defining

\[
\Psi_{\rho,R} : A_{\rho,R} \to \mathbb{R}, \quad \Psi_{\rho,R}(x) = \psi_{\rho,R}(r(x)).
\]

Here the extrinsic ball \( D_\rho(o) \) is as in Definition (2.3), and \( D_R(o) \) is that component of \( B_R(o) \cap P \) which contains \( D_\rho(o) \).

Then, the hypothesis (3.3) on the sectional curvatures, and the assumption (3.4) on the \( o \)-radial convexity lead to the following estimate using Proposition 2.11 (i) (recall that \( \psi'_{\rho,R}(r) \geq 0 \))

\[
\begin{align*}
\Delta^P \psi_{\rho,R}(r(x)) & \leq \left( \psi''_{\rho,R}(r(x)) - \psi'_{\rho,R}(r(x))\eta_w(r(x)) \right) g^2(r(x)) \\
& \quad + m \psi'_{\rho,R}(r(x)) (\eta_w(r(x)) - h(r(x))).
\end{align*}
\] (4.6)

So, using the second inequality in (4.5) and that \( \| \nabla^P r \| \geq g(r) \), we have:

\[
\begin{align*}
\Delta^P \psi_{\rho,R}(r(x)) & \leq \left( \psi''_{\rho,R}(r(x)) - \psi'_{\rho,R}(r(x))\eta_w(r(x)) \right) g^2(r(x)) \\
& \quad + m \psi'_{\rho,R}(r(x)) (\eta_w(r(x)) - h(r(x))) \\
& = g^2(r(x)) L_g \psi_{\rho,R}(r(x)) \\
& = 0 \\
& = \Delta^P v(x) ,
\end{align*}
\] (4.7)

where \( v(x) \) is the Laplace potential function for the extrinsic annulus \( A_{\rho,R} = D_R \setminus D_\rho \), setting \( v|_{\partial D_\rho} = 0 \) and \( v|_{\partial D_R} = 1 \).

Now, we apply the maximum principle to inequality (4.7) to obtain:

\[
\psi_{\rho,R}(r(x)) \geq v(x) , \text{ for all } x \in A_{\rho,R} .
\] (4.8)
This implies in particular that on $\partial D_\rho$ we have

\[(4.9) \quad \|\nabla^P \psi_{\rho,R}\| \geq \|\nabla^P v(x)\|_{\partial D_\rho}\|
\]

Then, using equation (2.13), we get

\[(4.10) \quad \text{Cap}(A_{\rho,R}) = \int_{\partial D_\rho} \|\nabla^P v(x)\| \, d\nu \\
\leq \int_{\partial D_\rho} \|\nabla^P \Psi_{\rho,R}\| \, d\mu \\
= \psi'_{\rho,R}(\rho) \int_{\partial D_\rho} \|\nabla^P r\| \, d\mu \\
= \frac{\text{Cap}_{Lg}(A_{\rho,R}^w)}{\text{Vol}(\partial D_{\rho}^w)} \int_{\partial D_\rho} \|\nabla^P r\| \, d\mu.
\]

On the other hand $D_\rho(o)$ is precompact with a smooth boundary and thence

\[(4.11) \quad \int_{\partial D_\rho} \|\nabla^P r\| \, d\mu > 0.
\]

Now, using equations (4.4) and (3.5) we have:

\[(4.12) \quad \text{Cap}(\bar{D}_\rho(o), P^m) = \lim_{R \to \infty} \text{Cap}(\bar{D}_\rho(o), D_R(o)) \\
\leq \left( \int_{\partial D_\rho} \|\nabla^P r\| \, d\mu \right) \left( \lim_{R \to \infty} \frac{\text{Cap}_{Lg}(A_{\rho,R}^w)}{\text{Vol}(\partial D_{\rho}^w)} \right) = 0
\]

Thus, $D_\rho(o)$ is a non-empty precompact subset with zero capacity in $P^m$, so the submanifold is parabolic.

**Assertion (A) in Theorem 3.6.**

Concerning the proof of assertion (A) in Theorem 3.6 and under the balance condition (3.2), we have that the solution of the problem (4.2) satisfies:
\[ \psi_{\rho,R}'(r) \geq 0 \]  
(4.13)

\[ \psi_{\rho,R}''(r) - \psi_{\rho,R}'(r)\eta_w(r) = -\psi_{\rho,R}'(r) \frac{\mathcal{M}(r)}{g^2(r)} \geq 0 \]

Then taking into account that \( \| \nabla^P r \|^2 \geq g \) and \( \phi_{\rho,R}''(r) - \phi_{\rho,R}'(r)\eta_w(r) \geq 0 \) we obtain, applying Proposition 2.11 (ii) to the transplanted solution \( \psi_{\rho,R} \),

\[ \Delta^P \psi_{\rho,R}(r(x)) \geq (\psi_{\rho,R}''(r(x)) - \psi_{\rho,R}'(r(x))\eta_w(r(x))) g^2(r(x)) \]
\[ + m\psi_{\rho,R}'(r(x)) (\eta_w(r(x)) - h(r(x))) \]
(4.14)
\[ = g^2(r(x)) L_\rho \psi_{\rho,R}(r(x)) \]
\[ = 0 \]
\[ = \Delta^P v(x) \, . \]

We now consider inequality (4.14), and proceed as in the proof of Theorem 3.4, but inverting the inequalities: applying the maximum principle, we have on \( \partial D_\rho \)

\[ \| \nabla^S \psi_{\rho,R} \| \leq \| \nabla^P v(x) \|_{\partial D_\rho} \]

and using equation (2.13), we get

\[ \text{Cap}(A_{\rho,R}) \geq \frac{\text{Cap}_L(A_{\rho,R}^w)}{\text{Vol}(\partial D_{\rho}^w)} \int_{\partial D_\rho} \| \nabla^S r \| d\mu. \]

Finally, taking into account that we have inequality (4.11), and that condition \( \int_{\rho}^{\infty} \Lambda_g(t) dt < \infty \) is satisfied, we obtain

(4.15) \[ \text{Cap}(\bar{D}_\rho(o), P^m) \geq \left( \int_{\partial D_\rho} \| \nabla^P r \| d\mu \right) \lim_{R \to \infty} \frac{\text{Cap}_L(A_{\rho,R}^w)}{\text{Vol}(\partial D_{\rho}^w)} > 0 \]

Thus, \( \bar{D}_\rho(o) \) is a compact subset with positive capacity in \( P^m \), and \( P \) is hyperbolic.
5. Proof of assertion (B) in Theorem 3.4 and Theorem 3.6

Let us define now the following second order differential operator \( L \):

\[
L \phi(r) = \phi''(r) + \phi'(r) (M(r) - \eta_w(r)).
\]

This is the same as in the proof above, with \( g(r) = 1 \ \forall r \). And consider now the smooth radial solution \( \phi_{\rho,R}(r) \) of the Dirichlet-Poisson problem associated to \( L \) and defined on the interval \( [\rho, R] \). As in the above proof (see Proposition 4.1), it is easy to check that

\[
\phi_{\rho,R}(r) = \frac{\int_{\rho}^{r} \Lambda(t) \, dt}{\int_{\rho}^{R} \Lambda(t) \, dt},
\]

where \( \Lambda(r) \) is the function introduced in Definition 3.2 (i).

The corresponding ‘drifted’ 2-capacity is

\[
\text{Cap}_L(A_{w}^{\rho,R}) = \int_{\partial D_{\rho}^{w}} \langle \nabla P \phi_{\rho,R}, \nu \rangle \, dA
\]

\[
= \text{Vol}(\partial D_{\rho}^{w}) \Lambda(\rho) \left( \int_{\rho}^{R} \Lambda(t) \, dt \right)^{-1}.
\]

Assertion (B) in Theorem 3.4.

Concerning assertion (B) in Theorem 3.4, we use equation (5.2) and the balance condition (3.2) to get

\[
\phi_{\rho,R}'(r) \geq 0
\]

\[
\phi_{\rho,R}''(r) - \phi_{\rho,R}'(r) \eta_w(r)
\]

\[
= -\phi_{\rho,R}'(r) M(r) \geq 0
\]

because \( M(r) \leq 0 \).
Taking into account that \( \| \nabla P_r \|^2 \leq 1 \) and \( \phi''_{\rho,R}(r) - \phi'_{\rho,R}(r)\eta_w(r) \geq 0 \) we obtain, applying Proposition 2.11 (i) to the transplanted function \( \phi_{\rho,R} \),

\[
\Delta^P \phi_{\rho,R}(r(x)) \leq \left( \phi''_{\rho,R}(r(x)) - \phi'_{\rho,R}(r(x))\eta_w(r(x)) \right) + m\phi'_{\rho,R}(r(x)) (\eta_w(r(x)) - h(r(x)))
\]

(5.5)

\[
= L \phi_{\rho,R}(r(x))
\]

\[
= 0
\]

\[
= \Delta^P v(x)
\]

In this last inequality, the function \( v(x) \) is the Laplace potential function for the extrinsic annulus \( A_{\rho,R} = D_R \setminus D_\rho \), setting \( v|_{\partial D_\rho} = 0 \) and \( v|_{\partial D_R} = 1 \).

Now the parabolicity of \( P \) follows as the proof of assertion (A) in Theorem 3.4. **Assertion (B) in Theorem 3.6.**

To show assertion (B) in Theorem 3.6, we consider the same second order differential operator \( L \), with the same smooth solution \( \phi_{\rho,R}(r) \) to the same Dirichlet-Poisson problem defined on the interval \([\rho, R]\).

But now we have

\[
\phi'_{\rho,R}(r) \geq 0
\]

(5.6)

\[
\phi''_{\rho,R}(r) - \phi'_{\rho,R}(r)\eta_w(r) = -\phi'_{\rho,R}(r)\mathcal{M}(r) \leq 0
\]

because \( \mathcal{M}(r) \geq 0 \).

Then, taking into account that \( \| \nabla P_r \|^2 \leq 1 \) and \( \phi''_{\rho,R}(r) - \phi'_{\rho,R}(r)\eta_w(r) \leq 0 \) we obtain, applying Proposition 2.11 (ii) to the transplanted function \( \phi_{\rho,R} \),

\[
\Delta^P \phi_{\rho,R}(r(x)) \geq \left( \phi''_{\rho,R}(r(x)) - \phi'_{\rho,R}(r(x))\eta_w(r(x)) \right) + m\phi'_{\rho,R}(r(x)) (\eta_w(r(x)) - h(r(x)))
\]

(5.7)

\[
= L \phi_{\rho,R}(r(x))
\]

\[
= 0
\]

\[
= \Delta^P v(x)
\].
Now the hyperbolicity of $P$ follows as the proof of assertion (A) of Theorem 3.6.

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