CHARACTERIZING GROUP C^*-ALGEBRAS THROUGH THEIR UNITARY GROUPS: THE ABELIAN CASE

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Abstract. We study to what extent group C^*-algebras are characterized by their unitary groups. A complete characterization of which Abelian group C^*-algebras have isomorphic unitary groups is obtained. We compare these results with other unitary-related invariants of $C^*(\Gamma)$, such as the K-theoretic $K_1(C^*(\Gamma))$ and find that C^*-algebras of nonisomorphic torsion-free groups may have isomorphic K_1-groups, in sharp contrast with the well-known fact that $C^*(\Gamma)$ (even Γ) is characterized by the topological group structure of its unitary group when Γ is torsion-free.

1. Introduction

The index theorem states that every continuous $f\colon \mathbb{T} \to \mathbb{T}$ is homotopic to the function $t \mapsto t^n$ for some $n \in \mathbb{Z}$ (its winding number). As a consequence the quotient of the unitary group of $C^*(\mathbb{Z})$ by its connected component is isomorphic to $\mathbb{Z}$. This identification can be extended in a functorial fashion to finitely generated Abelian groups and their inductive limits. Since every torsion-free Abelian group is an inductive limit of finitely generated groups, the following theorem, that we take as the departing point of our paper, follows.

Theorem 1.1 (see Theorem 8.57 of [8]). *If Γ is a torsion-free Abelian group the quotient $\mathcal{U}/\mathcal{U}_0$ of the unitary group $\mathcal{U} = \mathcal{U}(C^*(\Gamma))$ by its connected component $\mathcal{U}_0$ is isomorphic to Γ. Hence, two torsion-free Abelian groups Γ_1 and Γ_2 with topologically isomorphic unitary groups $\mathcal{U}(C^*(\Gamma_1))$ and $\mathcal{U}(C^*(\Gamma_2))$ must already be isomorphic.*

Another unitary-related invariant of $C^*(\Gamma)$ of great importance is the K_1-group, $K_1(C^*(\Gamma))$. Since $K_1(C^*(\mathbb{Z}^m)) = \mathbb{Z}^{2^{m-1}}$, two finitely generated groups are isomorphic whenever its K_1-groups are. The way this fact is proved does not however allow a functorial extension to inductive limits and, indeed, we construct in Section 3 two nonisomorphic torsion-free groups Γ_1 and Γ_2 with isomorphic K_1-groups, thereby showing that Theorem 1.1 is not valid for K_1-groups instead of unitary groups. We find therefore that $\mathcal{U}(C^*(\Gamma))$ is a stronger invariant than $K_1(C^*(\Gamma))$, for torsion-free Abelian groups. For general (even Abelian) groups this is no longer true, $K_1(C^*(\Gamma))$ distinguishes between groups with different finitely generated torsion-free quotients, while $\mathcal{U}(C^*(\Gamma))$ needs not, see Section 5.

With the above ideas as motivation we devote Section 4 to characterize when two Abelian groups Γ_1 and Γ_2 have isomorphic unitary groups. The groups $\mathcal{U}(C^*(\Gamma_1))$ and $\mathcal{U}(C^*(\Gamma_2))$ are shown to be topologically isomorphic if and only if $|\Gamma_1/t(\Gamma_1)| = |\Gamma_2/t(\Gamma_2)| =: \alpha$ and $\oplus_\alpha \Gamma_1/t(\Gamma_1)$ is group-isomorphic to $\oplus_\alpha \Gamma_2/t(\Gamma_2)$.

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We devote the last Section to derive some Examples from the above characterization.

2. Background

This paper is concerned with group C^*-algebras. The C^*-algebra $C^*(\Gamma)$ of a group Γ is defined as the enveloping C^*-algebra of the convolution algebra $L^1(\Gamma)$ and, as such, encodes the representation theory of Γ, see [3, Paragraph 13].

When Γ is a discrete Abelian group, $C^*(\Gamma)$ is a commutative C^*-algebra with spectrum homeomorphic to the compact group $\widehat{\Gamma}$, the group of characters of Γ. We may thus identify $C^*(\Gamma)$ with the algebra of continuous functions $C(\widehat{\Gamma},\mathbb{C})$ and the Gelfand transform coincides with the Fourier transform. The unitary group $\mathcal{U}(C^*(\Gamma))$ can therefore be identified with the topological group of $\mathbb{T}$-valued functions $C(\widehat{\Gamma},\mathbb{T})$.

We analyze in this paper to what extent a group Γ, or rather the C^*-algebra structure of $C^*(\Gamma)$, is determined by the topological group structure of $\mathcal{U}(C^*(\Gamma))$. For commutative Γ this amounts to asking to what extent Γ is determined by $C(\widehat{\Gamma},\mathbb{T})$.

The unitary groups $\mathcal{U}(C^*(\Gamma))$ are obviously related to another invariant of $C^*(\Gamma)$ of greater importance, the K_1-group of K-theory. K-theory for C^*-algebras is based on two functors, namely, K_0 and K_1, which associate to every C^*-algebra $\mathcal{A}$, two Abelian groups $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$. The group $K_1(\mathcal{A})$ is in particular defined by identifying unitary elements of *matrix algebras* over $\mathcal{A}$. It is allowing matrices over $\mathcal{A}$ (instead of elements of $\mathcal{A}$) what makes K_1-groups Abelian. When Γ is already Abelian, the determinant map $\Delta\colon \mathcal{U}(\mathcal{A})/\mathcal{U}(\mathcal{A})_0 \to K_1(\mathcal{A})$ is a *right inverse* of the canonical embedding $\omega\colon \mathcal{U}(\mathcal{A})/\mathcal{U}(\mathcal{A})_0 \to K_1(\mathcal{A})$ (see [12, Section 8.3]) and the link between $K_1(\mathcal{A})$ and $\mathcal{U}(\mathcal{A})$ is stronger.

The commonly used notation $K_*(\mathcal{A}) = K_1(\mathcal{A}) \oplus K_0(\mathcal{A})$ will also be adopted in this paper.

3. A torsion-free group Γ not determined by $K_1(C^*(\Gamma))$

As stated in the introduction, there is a group isomorphism $\mathrm{In}\colon C(\mathbb{T},\mathbb{T})/C(\mathbb{T},\mathbb{T})_0 \to \mathbb{Z}$ assigning to every $f \in C(\mathbb{T},\mathbb{T})$ its winding number. In other words, every element of $C(\mathbb{T},\mathbb{T})$ is homotopic to exactly one character of $\mathbb{T}$. This point of view can be carried over to $\mathbb{T}^n$ and then, taking projective limits, to every compact connected group, ultimately leading to Theorem 1.1, after identifying $C(\widehat{\Gamma},\mathbb{T})$ with $\mathcal{U}(C^*(\Gamma))$.

Despite the strong relation between $\mathcal{U}(C^*(\Gamma))$ and $K_1(C^*(\Gamma))$ we construct in this section two non-isomorphic torsion-free Abelian groups Γ_1 and Γ_2 with $K_1(\mathcal{U}(C^*(\Gamma_1)))$ isomorphic to $K_1(\mathcal{U}(C^*(\Gamma_2)))$.

3.1. **The structure of $K_1(C^*(\Gamma))$ for torsion-free Abelian Γ.** A countable torsion-free Abelian group Γ can always be obtained as the inductive limit of torsion-free finitely generated Abelian groups. Simply enumerate $\Gamma = \{\gamma_n \colon n < \omega\}$, define $\Gamma_n = \langle \gamma_j \colon 1 \leq j \leq n\rangle$ and let $\phi_n\colon \Gamma_n \to \Gamma_{n+1}$ define the inclusion mapping, then $\Gamma = \varinjlim(\Gamma_n, \phi_n)$. Each homomorphism ϕ_n then induces a morphism of C^*-algebras $\phi_n^*\colon C^*(\Gamma_n) \to C^*(\Gamma_{n+1})$, and $C^*(\Gamma) = \varinjlim(C^*(\Gamma_n), \phi_n^*)$

The functor K_1 commutes with inductive limits, see for instance [12]. If $K_1(\phi_n)\colon K_1(C^*(\Gamma_n)) \to K_1(C^*(\Gamma_{n+1}))$ denotes the homomorphism induced by the morphism ϕ_n^* , we have that

$$K_1(C^*(\Gamma)) = \varinjlim(K_1(C^*(\Gamma_n)), K_1(\phi_n)).$$

Since the groups Γ_n in the above discussion are all isomorphic to $\mathbb{Z}^{k(n)}$, for suitable $k(n)$, and $K_*(C^*(\mathbb{Z}^k))$ is isomorphic to the exterior product $\wedge\mathbb{Z}^k$, exterior products of Abelian groups provide concrete realizations of $K_1(C^*(\Gamma))$ that will be helpful in determining our examples.

Recall that the k-th. exterior, or wedge, product $\wedge^k(\mathbb{Z}^n)$ of a finitely generated group $\mathbb{Z}^n$ with free generators $e_1, \ldots, e_n$ is isomorphic to the free Abelian group generated by $\{e_{i_1}\wedge\cdots\wedge e_{i_k} : \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}\}$. A group homomorphism $\phi\colon \mathbb{Z}^n \to \mathbb{Z}^m$ induces a group homomorphism $\wedge^k(\phi)\colon \wedge^k(\mathbb{Z}^n) \to \wedge^k(\mathbb{Z}^m)$ in the obvious way $\wedge^k(\phi)(e_{i_1}\wedge\cdots\wedge e_{i_k}) = \phi(e_{i_1})\wedge\cdots\wedge\phi(e_{i_k})$. If $\Gamma = \varinjlim(\Gamma_i, h_i)$ is a direct limit, $\wedge^j(\Gamma)$ can be obtained as $\wedge^j(\Gamma) = \varinjlim(\wedge^j(\Gamma_i), \wedge^j(h_i))$. Other elementary properties of exterior products are best understood taken into account that $\wedge\Gamma$ is isomorphic to the quotient of $\bigotimes\Gamma$ by the twosided ideal N spanned by tensors of the form $g \otimes g$. The reference [1] is a classical one concerning exterior products.

The following result is well-known ([2, 5]), we supply a proof for the reader's convenience.

Lemma 3.1 ([5], Paragraph 2.1)**.** *Let Γ be a torsion-free discrete Abelian group. Then*

$$K_1(C^*(\Gamma)) \cong \wedge_{odd}\Gamma := \bigoplus_{j=0}^{\infty} \wedge^{2j+1}\Gamma.$$

Proof. Recall in first place that there is a unique ring isomorphism $R\colon \wedge\mathbb{Z}^n \to K_*(C^*(\mathbb{Z}^n))$ respecting the canonical embeddings of $\mathbb{Z}^n$ in both $K_*(C^*(\mathbb{Z}^n))$ and $\wedge\mathbb{Z}^n$. Since $K_*(C^*(\mathbb{Z}^n)) = K_0(C^*(\mathbb{Z}^n)) \oplus K_1(C^*(\mathbb{Z}^n))$ and the ring structure $K_*(C^*(\mathbb{Z}^n))$ is $\mathbb{Z}_2$-graded (which means that $x \in K_i(C^*(\mathbb{Z}^n))$, $y \in K_j(C^*(\mathbb{Z}^n))$ implies that $xy \in K_{i+j}(C^*(\mathbb{Z}^n))$ with $i, j \in \mathbb{Z}_2$), we have that the isomorphism R maps $\wedge_{\mathrm{odd}}\mathbb{Z}^n$ onto $K_1(C^*(\mathbb{Z}^n))$.

Now put $\Gamma = \varinjlim(\Gamma_n, \phi_n)$ with $\Gamma_n \cong \mathbb{Z}^{j_n}$. The uniqueness of the above mentioned ring-isomorphism, together with the fact that wedge products commute with direct limits implies that $K_1(C^*(\Gamma))$ is isomorphic to $\wedge_{\mathrm{odd}}\Gamma$. $\square$

Since the groups Γ_n are always isomorphic to $\mathbb{Z}^{k(n)}$ a comparison between Γ and $K_1(C^*(\Gamma))$ turns into a comparison of two inductive limits, $\varinjlim(\mathbb{Z}^{k(n)}, \phi_n)$ and $\varinjlim(\mathbb{Z}^{2^{k(n)-1}}, K_1(\phi_n))$. When Γ has finite rank m it may be assumed without loss of generality that $k(n) = m$ for all n. If in addition $m \leq 2$, it is easy to see (cf. Lemma 3.5) that $K_1(\phi_n) = \phi_n$. We have thus:

Corollary 3.2. *If Γ is a torsion-free Abelian group of rank ≤ 2, then $K_1(C^*(\Gamma))$ is isomorphic to Γ.*

Corollary 3.2 shows that two nonisomorphic torsion-free Abelian groups Γ_i with $K_1(C^*(\Gamma_1))$ isomorphic to $K_1(C^*(\Gamma_2))$ must have rank greater than 2. For our counterexample we will deal with two groups of rank 4. If Γ is such a group, then $K_1(C^*(\Gamma))$ is isomorphic to $\wedge^1(\Gamma) \oplus \wedge^3(\Gamma)$. Our selection of the examples is determined by the following theorem of Fuchs and Loonstra.

Theorem 3.3 (Particular case of Theorem 90.3 of [6])**.** *There are two non-isomorphic groups Γ_1 and Γ_2, both of rank 2, such that*

$$\Gamma_1 \oplus \Gamma_1 \cong \Gamma_2 \oplus \Gamma_2.$$

We then have:

Theorem 3.4. *Let Γ_1, Γ_2 be the groups of Theorem 3.3 and define the 4-rank groups, $\Delta_i = \mathbb{Z} \oplus \mathbb{Z} \oplus \Gamma_i$. Then $K_1(C^*(\Delta_1))$ and $K_1(C^*(\Delta_2))$ are isomorphic, while Δ_1 and Δ_2 are not.*

We shall split the proof of Theorem 3.4 in several Lemmas. We begin by observing how Lemma 3.1 makes the groups $K_1(C^*(\Delta_i))$ easily realizable.

Lemma 3.5. *If Γ is a torsion-free Abelian group of rank 2 and $\Delta = \mathbb{Z} \oplus \mathbb{Z} \oplus \Gamma$, then*

$$K_1(C^*(\Delta)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \Gamma \oplus \Gamma \oplus \wedge^2\Gamma \oplus \wedge^2\Gamma.$$

Proof. As Δ has rank 4,

$$\wedge_{\mathrm{odd}}\Delta = \wedge^1\Delta \oplus \wedge^3\Delta \cong \Delta \oplus \wedge^3\Delta. \tag{1}$$

Put $\Gamma = \varinjlim(\Gamma_n, \phi_n)$, with $\Gamma_n \cong \mathbb{Z}^2$. Then, defining $\mathrm{id} \oplus \mathrm{id} \oplus \phi_n \colon \mathbb{Z} \oplus \mathbb{Z} \oplus \Gamma_n \to \mathbb{Z} \oplus \mathbb{Z} \oplus \Gamma_{n+1}$ in the obvious way, we have that $\Delta = \varinjlim(\mathbb{Z} \oplus \mathbb{Z} \oplus \Gamma_n, \mathrm{id} \oplus \mathrm{id} \oplus \phi_n)$ and $\wedge^3\Delta = \varinjlim(\wedge^3(\mathbb{Z} \oplus \mathbb{Z} \oplus \Gamma_n), \wedge^3(\mathrm{id} \oplus \mathrm{id} \oplus \phi_n))$.

If e_j^n, $j = 1, 2$ are the generators of $\mathbb{Z} \oplus \mathbb{Z}$ and f_j^n, $j = 1, 2$ are the generators of Γ_n, $\wedge^3(\mathbb{Z} \oplus \mathbb{Z} \oplus \Gamma_n) = \langle e_1^n \wedge e_2^n \wedge f_1^n, e_1^n \wedge e_2^n \wedge f_2^n, e_1^n \wedge f_1^n \wedge f_2^n, e_2^n \wedge f_1^n \wedge f_2^n \rangle$. The images of each of these generators under the homomorphism $\wedge^3(\mathrm{id} \oplus \mathrm{id} \oplus \phi_n)$ are:

$$\wedge^3(\mathrm{id} \oplus \mathrm{id} \oplus \phi_n)\Big(e_1^n \wedge e_2^n \wedge f_j^n\Big) = e_1^{n+1} \wedge e_2^{n+1} \wedge \phi_n(f_j^n), \quad j = 1, 2$$
$$\wedge^3(\mathrm{id} \oplus \mathrm{id} \oplus \phi_n)\Big(e_j^n \wedge f_1^n \wedge f_2^n\Big) = e_j^{n+1} \wedge \big(\wedge^2(\phi_n)(f_1^n \wedge f_2^n)\big), \quad j = 1, 2.$$

In the limit, the thread formed by the first two generators will yield a copy of Γ while the one formed by each of the other two will yield a copy of $\wedge^2\Gamma$. This and (1) give the Lemma. $\square$

We now take care of $\wedge^2(\Gamma)$. This is a rank one group. Abelian groups of rank one are completely classified by their so-called type.

The type of an Abelian group A is defined in terms of p-heights. Given a prime p, the largest integer k such that $p^k \mid a$ is called the *p-height* $h_p(a)$ of A. The sequence of p-heights $\chi(a) = (h_{p_1}(a), \ldots, h_{p_n}(a), \ldots)$ is then called the *characteristic* or the *height-sequence* of a. Two characteristics $(k_1, \ldots, k_n, \ldots)$ and $(l_1 \ldots, l_n, \ldots)$ are considered equivalent if $k_n = l_n$ for all but a finite number of finite indices. An equivalence class of characteristics is called a *type*. If $\chi(a)$ belongs to a type $\mathbf{t}$, then we say that *a is of type* $\mathbf{t}$. In a torsion-free group of rank one all elements are of the same type (such groups are called *homogeneous*). For more details about p-heights, types..., see [6]. The only fact we need here is that two groups of rank 1 are isomorphic if and only if they have nontrivial elements with the same type, Theorem 85.1 of [6].

We now study the type of groups $\Gamma \wedge \Gamma$ with Γ of rank 2.

Lemma 3.6. *Let Γ be a torsion-free group of rank 2 and let $x_1, x_2 \in \Gamma$. The element $x_1 \wedge x_2 \in \Gamma \wedge \Gamma$ is divisible by the integer m if and only if there is some element $k_1x_1 + k_2x_2 \in \Gamma$ divisible by m with either k_1 or k_2 coprime with m.*

Proof. We can without loss of generality assume that the subgroup generated by x_1, x_2 is isomorphic to $\mathbb{Z}^2$ and that Γ is an additive subgroup of the vector space spanned over $\mathbb{Q}$ by x_1, x_2. Now $x \wedge y$ will be divisible by m if and only if there are elements u_1, u_2 in Γ such that $u_i = \alpha_{i1}x_1 + \alpha_{i2}x_2$ with $\det(\alpha_{ij}) = 1/m$ (note that $u_1 \wedge u_2 = \det(\alpha_{ij})\, x_1 \wedge x_2$). To get that determinant we clearly need some denominator m and we can assume (by conveniently modifying the α_{ij}'s) that $\alpha_{11} = k_1/m$ and $\alpha_{12} = k_2/m$ with either k_1 or k_2 coprime with m. The element of Γ we were seeking is then $k_1x_1 + k_2x_2$. □

Lemma 3.7. *Let Γ_1 and Γ_2 be two rank 1, torsion-free Abelian groups. If $\Gamma_1 \oplus \Gamma_1 \cong \Gamma_2 \oplus \Gamma_2$, then $\wedge^2(\Gamma_1) \cong \wedge^2(\Gamma_2)$.*

Proof. Let $\{v_1, w_1\}$ and $\{v_2, w_2\}$ be maximal independent sets in Γ_1 and Γ_2, respectively.

By conveniently re-defining the elements v_i and w_i it may be assumed that

$$\phi(v_1, 0) = (\alpha_{11}v_2 + \alpha_{12}w_2, \beta_{11}v_2 + \beta_{12}w_2)$$
$$\phi(w_1, 0) = (\alpha_{21}v_2 + \alpha_{22}w_2, \beta_{21}v_2 + \beta_{22}w_2),$$

with $\alpha_{ij}, \beta_{i,j} \in \mathbb{Z}$, $i, j \in \{1, 2\}$.

We will now find a finite set of primes F such that $v_2 \wedge w_2$ is divisible by p^k whenever $v_1 \wedge w_1$ is divisible by p^k, for every prime $p \notin F$. Since the whole process can be repeated for ϕ^{-1}, this will show that $v_1 \wedge w_1$ and $v_2 \wedge w_2$ have the same type.

Since ϕ is an isomorphism, the matrix

$$M = \begin{pmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \\ \beta_{11} & \beta_{21} \\ \beta_{12} & \beta_{22} \end{pmatrix}$$

has rank two. At least one of the following submatrices must then have rank 2 as well:

$$M_1 = \begin{pmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{pmatrix}, \quad M_2 = \begin{pmatrix} \beta_{11} & \beta_{21} \\ \beta_{12} & \beta_{22} \end{pmatrix} \quad \text{or} \quad M_3 = \begin{pmatrix} \alpha_{11} & \alpha_{21} \\ \beta_{11} & \beta_{21} \end{pmatrix}.$$

Let p be any prime not dividing $\det(M_1)$, $\det(M_2)$ or $\det(M_3)$ and suppose p^k divides $v_1 \wedge w_1$. By Lemma 3.6 there is an element $A = k_1v_1 + k_2w_1 \in \Gamma_1$ divisible by p^k with either k_1 or k_2 coprime with p. Then

(2)

$$\phi(A, 0) = k_1\phi(v_1, 0) + k_2\phi(w_1, 0) =$$
$$\Big((k_1\alpha_{11} + k_2\alpha_{21})v_2 + (k_1\alpha_{12} + k_2\alpha_{22})w_2\,,\ (k_1\beta_{11} + k_2\beta_{21})v_2 + (k_1\beta_{12} + k_2\beta_{22})w_2\Big) \in \Gamma_2 \times \Gamma_2$$

Suppose for instance that M_1 has rank 2. The only solution modulo p to the system

$$\begin{cases} \alpha_{11}x + \alpha_{21}y &= 0 \\ \alpha_{12}x + \alpha_{22}y &= 0 \end{cases}$$

is then the trivial one. The integers k_1 and k_2 cannot therefore be a solution to the system (they are not both coprime with p). It follows that one of the integers $k_1\alpha_{11} + k_2\alpha_{21}$ or $\alpha_{12}k_1 + \alpha_{22}k_2$ is not a multiple of p.

If M_2 or M_3 have rank two we argue exactly in the same way. At the end we find that at least one of the $k_1\alpha_{1i} + k_2\alpha_{2i}$ or $k_1\beta_{1i} + k_2\beta_{2i}$ is not a multiple of p .

We know by (2) that both $(k_1\alpha_{11} + k_2\alpha_{21})v_2 + (k_1\alpha_{12} + k_2\alpha_{22})w_2$ and $(k_1\beta_{11} + k_2\beta_{21})v_2 + (k_1\beta_{12} + k_2\beta_{22})w_2$ are divisible by p^k and we conclude with Lemma 3.6 that $v_2\wedge w_2$ is divisible by p^k. □

Proof of Theorem 3.4 To see that $K_1(C^*(\Delta_1)) \cong K_1(C^*(\Delta_2))$, simply put together Lemma 3.7 and Lemma 3.5.

Since Γ_1 and Γ_2 are not isomorphic and finitely generated Abelian groups have the cancellation property, Δ_1 and Δ_2 cannot be isomorphic, either.

Remark 3.8. The argument of Lemma 3.5 shows that $K_0(C^*(\Delta))$ is (again) isomorphic to $\mathbb{Z}\oplus\mathbb{Z}\oplus\Gamma\oplus\Gamma\oplus\wedge^2(\Gamma)\oplus\wedge^2(\Gamma)$ (this time $K_0(C^*(\Delta)) \cong \wedge^0\Delta\oplus\wedge^2\Delta\oplus\wedge^4\Delta$ with $\wedge^2\Delta \cong \mathbb{Z}\oplus\Gamma\oplus\Gamma\oplus\wedge^2\Gamma$ and $\wedge^4\Delta \cong \wedge^2\Gamma$.

The group C^*-algebras $C^*(\Delta_1)$ and $C^*(\Delta_2)$ of Theorem 3.4 have therefore the same K-theory.

4. Relating $\mathcal{U}(C^*(\Gamma))$ and Γ

This Section is devoted to evidence what is the relation between two C^*-algebras $C^*(\Gamma_1)$ and $C^*(\Gamma_2)$ with topologically isomorphic unitary groups. A result like Theorem 1.1 cannot be expected for general Abelian groups, as for instance all countably infinite torsion groups have isometric C^*-algebras. The right question to ask is obviously whether group C^*-algebras are determined by their unitary groups. Even if this question also has a negative answer, two group C^*-algebras $C^*(\Gamma_1)$ and $C^*(\Gamma_2)$ are strongly related when $\mathcal{U}(C^*(\Gamma_1))$ and $\mathcal{U}(C^*(\Gamma_2))$ are topologically isomorphic as the contents of this Section show. Our main tools here will be of topological nature and we shall regard $\mathcal{U}(C^*(\Gamma))$ as $C(\widehat{\Gamma},\mathbb{T})$.

We begin with a well-known observation. Denote by $C^0(X,\mathbb{T})$ the subgroup of $C(X,\mathbb{T})$ consisting of all nullhomotopic maps, that is, $C^0(X,\mathbb{T})$ is the connected component of the identity of $C(X,\mathbb{T})$. Let also $\pi^1(X)$ denote the quotient $C(X,\mathbb{T})/C^0(X,\mathbb{T})$, also known as the first cohomotopy group of X and often denoted as $[X,\mathbb{T}]$. It is well known that $C^0(X,\mathbb{T})$ coincides with the group of functions that factor through $\mathbb{R}$, that is, $C^0(X,\mathbb{T})$ is the range of the exponential map $\exp : C(X,\mathbb{R}) \to C(X,\mathbb{T})$.

Lemma 4.1. *If X is a compact Hausdorff space, the structure of $C(X,\mathbb{T})$ is described by the following exact sequence*

$$0 \to C(X,\mathbb{Z}) \to C(X,\mathbb{R}) \to C^0(X,\mathbb{T}) \to C(X,\mathbb{T}) \to \pi^1(X).$$

In addition $C^0(X,\mathbb{T})$ is open and splits, i.e., $C(X,\mathbb{T}) \cong C^0(X,\mathbb{T}) \oplus \pi^1(X)$.

Our second observation is that, as far as group C^*-algebras is concerned, all Abelian groups have a splitting torsion subgroup.

Theorem 4.2 (Corollary 10.38 [8]). *The connected component K_0 of a compact group K, splits topologically, i.e, K is homeomorphic to $K_0 \oplus K/K_0$.*

The character group of a countable discrete group Γ is a compact metrizable group $\widehat{\Gamma}$ and the set of characters that vanish on $t\Gamma$ coincides with the connected component of $\widehat{\Gamma}$, in symbols $t\Gamma^\perp = \widehat{\Gamma}_0$. Further, the duality between discrete Abelian and compact Abelian groups identifies $\widehat{t\Gamma}$ with the quotient $\widehat{\Gamma}/\widehat{\Gamma}_0$. It follows therefore from Theorem 4.2 that

$$\widehat{\Gamma} \sim \widehat{t\Gamma} \times (t\Gamma)^\perp \tag{3}$$

and, hence, that $C^*(\Gamma)$ is isometric to $C^*(t\Gamma \oplus \Gamma/t\Gamma)$.

We now turn our attention to groups with splitting connected component.

4.1. **The structure of unitary groups of certain commutative C^*-algebras.** We begin by noting that the additive structure of a commutative C^*-algebra contains very little information on the algebra. This fact will be useful in classifying unitary groups.

Theorem 4.3 (Milutin, see for instance Theorem III.D.18 of [13])**.** *If K_1 and K_2 are uncountable, compact metric spaces, then the Banach spaces $C(K_1,\mathbb{C})$ and $C(K_2,\mathbb{C})$ are topologically isomorphic.*

Lemma 4.4. *Let K and D be compact topological spaces, K connected and D totally disconnected. The following topological isomorphism then holds:*

$$C(K \times D, \mathbb{T}) \cong C(K \times D, \mathbb{R}) \times C(D, \mathbb{T}) \times \oplus_{w(D)} \pi^1(K), \tag{4}$$

where $w(D)$ denotes the topological weight of D.

Proof. We first observe that $C(K \times D, \mathbb{T})$ is topologically isomorphic to $C(D, C(K, \mathbb{T}))$. From Lemma 4.1 we deduce that

$$C(K \times D, \mathbb{T}) \cong C(D, C^0(K, \mathbb{T})) \times C(D, \pi^1(K)). \tag{5}$$

There is a topological isomorphism from the Banach space $C(K,\mathbb{R})$ onto the Banach space $C_\bullet(K,\mathbb{R})$ of functions sending 0 to 0. It is now easy to check that the mapping $(f,t) \mapsto t \cdot \exp(f)$ identifies $C_\bullet(K,\mathbb{R}) \times \mathbb{T}$ with $C^0(K,\mathbb{T})$ and hence

$$C^0(K,\mathbb{T}) \cong C(K,\mathbb{R}) \times \mathbb{T}.$$

Along with (5) we obtain

$$C(K\times D,\mathbb{T}) \cong C(D, C(K,\mathbb{R})\times\mathbb{T})\times C(D,\pi^1(K)) = C(D\times K,\mathbb{R})\times C(D,\mathbb{T})\times C(D,\pi^1(K)).$$

Now $\pi^1(K)$ is a discrete group and each element of $C(D,\pi^1(K))$ determines an open and closed subset of D. An analysis identical to that of [4] for $C(X,\mathbb{Z})$ then yields

$$C(D,\pi^1(K)) \cong \oplus_{w(D)} \pi^1(K),$$

and the proof follows. $\square$

The following lemma can be found as an exercise in [8].

Lemma 4.5. *If D is a totally disconnected compact space, $C(D,\mathbb{T}) = C^0(D,\mathbb{T})$ and $C(D,\mathbb{T})$ is connected.*

Theorem 4.6. *Let $X = K_1 \times D_1$ and $Y = K_2 \times D_2$ be two compact metrizable spaces with K_i connected and D_i totally disconnected for $i \in \{1,2\}$. The following assertions are then equivalent.*

(1) $C(X,\mathbb{T}) \cong C(Y,\mathbb{T})$.

(2) (a) $\bigoplus_{w(D_1)} \pi^1(K_1) \cong \bigoplus_{w(D_2)} \pi^1(K_2)$, *where* $w(D_1)$ *and* $w(D_2)$ *are the topological weights of* D_1 *and* D_2*, respectively, and*
(b) $C(D_1, \mathbb{T}) \cong C(D_2, \mathbb{T})$.

Proof. It is obvious from Theorem 4.3 (observe that $K_i \times D_i$ is uncountable as soon as K_i is nontrivial) and Lemma 4.4 that (2) implies (1).

We now use the decomposition of Lemma 4.4 to deduce (2) from (1). Assertion (a) follows from factoring out connected components in (4) (note that $C(K_i \times D_i, \mathbb{R}) \times C(D_i, \mathbb{T})$ is connected, use Lemma 4.5 for $C(D_i, \mathbb{T})$). The connected components $C(K_1 \times D_1, \mathbb{R}) \times C(D_1, \mathbb{T})$ and $C(K_2 \times D_2, \mathbb{R}) \times C(D_2, \mathbb{T})$ will be topologically isomorphic as well. Let $H\colon C(K_1 \times D_1, \mathbb{R}) \times C(D_1, \mathbb{T}) \to C(K_2 \times D_2, \mathbb{R}) \times C(D_2, \mathbb{T})$ denote this isomorphism.

Consider now the homomorphism $\widehat{H}\colon C(K_2 \times D_2, \mathbb{R})^\frown \times C(D_2, \mathbb{T})^\frown \to C(K_1 \times D_1, \mathbb{R})^\frown \times C(D_1, \mathbb{T})^\frown$ that results from dualizing H.

When D is a totally disconnected compact group, the only continuous characters of $C(D, \mathbb{T})$ are linear combinations with coefficients in $\mathbb{Z}$ of evaluations of elements of D, i.e., the group $C(D, \mathbb{T})^\frown$ is isomorphic to the free Abelian group on D [11] (see [7] for more on the duality between $C(X, \mathbb{T})$ and $A(X)$ based on the exact sequence in Lemma 4.1)).

There is on the other hand a well-known isomorphism between $C(K_1 \times D_1, \mathbb{R})^\frown$ and the vector space of all continuous linear functionals on $C(K_1 \times D_1, \mathbb{R})$, $C(K_1 \times D_1, \mathbb{R})^\frown$ is therefore a divisible group.

Since the groups $A(D_i)$ do not contain any divisible subgroup, $\widehat{H}(C(K_1 \times D_1, \mathbb{R})^\frown$ must equal $C(K_2 \times D_1, \mathbb{R})^\frown$. We deduce thus, taking quotients, that $C(D_1, \mathbb{T})$ and $C(D_2, \mathbb{T})$ are topologically isomorphic. □

4.2. **The group case.** We now specialize the results in the previous paragraphs for the case of a compact Abelian group.

When T is a torsion discrete Abelian group, $\widehat{T}$ is a compact totally disconnected group and hence homeomorphic to the Cantor set. The group C^*-algebras of all countably infinite torsion Abelian groups will therefore be isometric. These facts are summarized in the following lemma.

Lemma 4.7. *Let* T_1 *and* T_2 *be countable torsion discrete Abelian groups. Then the following assertions are equivalent:*

(1) *The group* C^**-algebras* $C^*(T_1)$ *and* $C^*(T_2)$ *are isomorphic as* C^**-algebras.*
(2) *The unitary groups of* $C^*(T_1)$ *and* $C^*(T_2)$ *are topologically isomorphic.*
(3) *The compact groups* $\widehat{T_1}$ *and* $\widehat{T_2}$ *are homeomorphic.*
(4) *The groups* T_1 *and* T_2 *have the same cardinal.*

Hence, the main result asserts:

Theorem 4.8. *Let* Γ_1 *and* Γ_2 *be countable discrete groups. The following are equivalent:*

(1) *The unitary groups of* $C^*(\Gamma_1)$ *and* $C^*(\Gamma_2)$ *are topologically isomorphic.*
(2) $|t\Gamma_1| = |t\Gamma_2| = \alpha$ *and*

$$\bigoplus_{\alpha} \frac{\Gamma_1}{t\Gamma_1} \cong \bigoplus_{\alpha} \frac{\Gamma_2}{t\Gamma_2}.$$

Proof. By (3) and Lemma 4.4

(6)
$$\mathcal{U}(C^*(\Gamma_i)) \cong C(\widehat{t\Gamma_i}\times(t\Gamma_i)^{\perp},\mathbb{T}) \cong C(\widehat{t\Gamma_i}\times(t\Gamma_i)^{\perp},\mathbb{R})\times C(\widehat{t\Gamma_i},\mathbb{T})\times \bigoplus_{w(\widehat{t\Gamma_i})} \pi^1((t\Gamma_i)^{\perp}),$$

where $(t\Gamma_i)^{\perp}$ are compact connected and $\widehat{t\Gamma_i}$ are compact totally disconnected Abelian groups.

Suppose first that $\mathcal{U}(C^*(\Gamma_1))$ and $\mathcal{U}(C^*(\Gamma_2))$ are topologically isomorphic. By Theorem 4.6, $C(\widehat{t\Gamma_1},\mathbb{T})$ is topologically isomorphic to $C(\widehat{t\Gamma_2},\mathbb{T})$. It follows from Lemma 4.7 that $\widehat{t\Gamma_1}$ and $\widehat{t\Gamma_2}$ are homeomorphic. Let $\alpha = w(\widehat{t\Gamma_1})$. By statement (a) of Theorem 4.6,
$$\bigoplus_{\alpha}\pi^1((t\Gamma_1)^{\perp}) \cong \bigoplus_{\alpha}\pi^1((t\Gamma_2)^{\perp}),$$
Now $\pi^1(t\Gamma_i^{\perp})$ is isomorphic by Theorem 1.1 to the torsion-free group $\Gamma_i/t(\Gamma_i)$. The above isomorphism thus becomes

(7)
$$\bigoplus_{\alpha}\left(\frac{\Gamma_1}{t\Gamma_1}\right) \cong \bigoplus_{\alpha}\left(\frac{\Gamma_2}{t\Gamma_2}\right)$$

and we are done.

Suppose conversely that assertion (2) holds. We have then from Lemma 4.7 that $C(\widehat{t\Gamma_1},\mathbb{T})$ and $C(\widehat{t\Gamma_1},\mathbb{T})$ are topologically isomorphic.

On the other hand, the isomorphism $\bigoplus_{\alpha}\frac{\Gamma_1}{t\Gamma_1} \cong \bigoplus_{\alpha}\frac{\Gamma_2}{t\Gamma_2}$ implies, by way of Theorem 1.1, that $\oplus_{\alpha}\pi^1((t\Gamma_1)^{\perp})$ is isomorphic to $\oplus_{\alpha}\pi^1((t\Gamma_2)^{\perp})$.

It follows then from Theorem 4.6 that $C(\widehat{\Gamma_1},\mathbb{T})$ and $C(\widehat{\Gamma_2},\mathbb{T})$, that is $\mathcal{U}(C^*(\Gamma_1))$ and $\mathcal{U}(C^*(\Gamma_2))$, are topologically isomorphic. □

5. Concluding remarks

Theorem 1.1 shows how strongly the topological group structure of $\mathcal{U}(\mathcal{A})$ may happen to determine a C^*-algebra $\mathcal{A}$. Theorem 4.8 then precises the amount of information on $\mathcal{A}$ that is encoded in $\mathcal{U}(\mathcal{A})$, for the case of a group C^*-algebra. This reveals some limitations on the strength of $\mathcal{U}(\mathcal{A})$ as an invariant of $\mathcal{A}$ that will be made concrete in this Section.

From Theorem 1.1 and Lemma 4.7 we have that $C^*(\Gamma)$ is completely determined by its unitary group when Γ is either torsion-free or a torsion group. This is not the case if Γ is a mixed group.

Example 5.1. *Two non-isometric group C^*-algebras with topologically isomorphic unitary groups.*

Proof. Let Γ_1 and Γ_2 be the groups in Theorem 3.3. Define $\Delta_i = \Gamma_i \oplus \mathbb{Z}_2$. Identifying as usual $C(\Delta_i,\mathbb{T})$ with $\mathcal{U}(C^*(\Delta_i))$ and applying Lemma 4.4, we have that
$$\mathcal{U}(C^*(\Delta_i)) \cong C(\Delta_i,\mathbb{R})\times\mathbb{T}^2\times(\Gamma_i\oplus\Gamma_i).$$
The election of Γ_i and Milutin's theorem show that $\mathcal{U}(C^*(\Delta_1))$ is topologically isomorphic to $\mathcal{U}(C^*(\Delta_2))$.

The algebras $C^*(\Delta_1)$ and $C^*(\Delta_2)$ are not isometric, since their spectra, $\widehat{\Gamma_1}\times\mathbb{Z}_2$ and $\widehat{\Gamma_2}\times\mathbb{Z}_2$, are not homeomorphic (their connected components are not homeomorphic). □

This example also shows that simple "duplications" of torsion-free groups are not determined by the unitary groups of their C^*-algebras:

Example 5.2. *Two non-isomorphic torsion-free groups* Γ_1 *and* Γ_2 *such that* $\mathcal{U}(C^*(\Gamma_1 \oplus \mathbb{Z}_2))$ *and* $\mathcal{U}(C^*(\Gamma_2 \oplus \mathbb{Z}_2))$ *are topologically isomorphic.*

Finally,

Example 5.3. *Two Abelian groups* Γ_1 *and* Γ_2 *of different torsion-free rank with* $\mathcal{U}(C^*(\Gamma_1))$ *topologically isomorphic to* $\mathcal{U}(C^*(\Gamma_2))$.

Proof. Let $\Gamma_1 = \mathbb{Z} \oplus (\oplus_\omega \mathbb{Z}_2)$ and $\Gamma_2 = (\mathbb{Z} \oplus \mathbb{Z}) \oplus (\oplus_\omega \mathbb{Z}_2)$. The argument now is as in Example 5.1. □

In the above example one can obviously replace Γ_2 by $(\oplus_\omega \mathbb{Z}) \oplus (\oplus_\omega \mathbb{Z}_2)$ and have an example of two Abelian groups with $\mathcal{U}(C^*(\Gamma_1))$ topologically isomorphic to $\mathcal{U}(C^*(\Gamma_2))$ while the torsion-free rank of one of them is finite and the torsion-free rank of the other is infinite.

5.1. **Invariants.** The unitary group $\mathcal{U}(C^*(\Gamma))$ is an invariant of the group $C^*(\Gamma)$, and as such can be compared with other well known unitary-related invariants, like for instance $K_1(C^*(\Gamma))$. We can also mention here related work of Hofmann and Morris on free compact Abelian groups [9]. This is part of a more general project of attaching a compact topological group $FC(X)$ to every compact Hausdorff space X. The free compact Abelian group on X is constructed as the character group of the *discrete* group $C(X, \mathbb{T})_d$. For an Abelian group Γ, this process produces an invariant of $C^*(\Gamma)$, namely the group $\mathcal{U}(C^*(\Gamma))_d$ equipped with the discrete topology. The character group of $\mathcal{U}(C^*(\Gamma))_d$ is precisely the free compact Abelian group on $\widehat{\Gamma}$. Being the same object but with no topology, this invariant is weaker than $\mathcal{U}(C^*(\Gamma))$. It is easy to see that it is indeed strictly weaker, simply take $\Gamma_1 = \mathbb{Q}$ and $\Gamma_2 = \oplus_\omega \mathbb{Q}$. In general there is a copy of the free Abelian group generated by X, densely embedded in $FC(X)$, $FC(X)$ is, actually (a realization of) the Bohr compactification of the free Abelian topological group on X (see [7] for detailed references on free Abelian topological groups and their duality properties). Since two topological spaces with topological isomorphic free Abelian topological groups must have the same covering dimension [10], Example 5.2 is somewhat unexpected.

The comparison with $K_1(\mathcal{U}(C^*(\Gamma)))$ is richer. As we saw in Section 3, the group algebras $C^*(\Gamma_1)$ and $C^*(\Gamma_2)$ of two non-isomorphic torsion-free Abelian groups Γ_1 and Γ_2 can have isomorphic K_1-groups, while their unitary groups must be topologically isomorphic by Theorem 1.1. The opposite direction does not work either. We find next two discrete groups whose group C^*-algebras have isomorphic unitary groups while their K_1-groups fail to be so. We first see that from Theorem 4.2 and with a simple application of the Künneth theorem, the K_1-group of a group C^*-algebra depends exclusively on its torsion-free component.

Lemma 5.4. *Let* Γ *be an Abelian discrete group. Then*

$$K_1(C^*(\Gamma)) \cong K_1(C^*(\Gamma/t\Gamma))$$

Proof. >From Theorem 4.2, $\widehat{\Gamma}$ is homeomorphic to $\widehat{\Gamma}/\widehat{\Gamma}_0 \times \widehat{\Gamma}_0$, where $\widehat{\Gamma}/\widehat{\Gamma}_0 \cong t\widehat{\Gamma}$ and $\widehat{\Gamma}_0 \cong t\Gamma^\perp \cong \widehat{\Gamma/t\Gamma}$. Therefore,

$$C^*(\Gamma) \cong C^*(t\Gamma) \otimes C^*(\Gamma/t\Gamma). \tag{8}$$

Applying the Künneth formula to (8), we obtain,

$$\begin{aligned} K_1(C^*(\Gamma)) &\cong K_1(C^*(t\Gamma)\otimes C^*(\Gamma/t\Gamma)) \\ &\cong K_0(C^*(t\Gamma))\otimes K_1(C^*(\Gamma/t\Gamma))\oplus K_1(C^*(t\Gamma))\otimes K_0(C^*(\Gamma/t\Gamma)) \\ &\cong \mathbb{Z}\otimes K_1(C^*(\Gamma/t\Gamma))\cong K_1(C^*(\Gamma/t\Gamma)), \end{aligned}$$

since $K_0(C(D)) = \mathbb{Z}$ and $K_1(C(D)) = 0$ for a infinite totally disconnected compact group D. □

Example 5.5. *Two Abelian groups Γ_1 and Γ_2 whose group C^*-algebras have topologically isomorphic unitary groups, whereas their K_1-groups are non-isomorphic.*

Proof. Take Γ_1 and Γ_2 from Example 5.3. Applying Lemma 5.4 and Lemma 3.1, we have that

$$K_1(C^*(\Gamma_1)) \cong K_1(C^*(\mathbb{Z})) \cong \mathbb{Z} \quad \text{and} \quad K_1(C^*(\Gamma_2)) \cong K_1(C^*(\mathbb{Z}\oplus\mathbb{Z})) \cong \mathbb{Z}\oplus\mathbb{Z}.$$

The topological groups $\mathcal{U}(C^*(\Gamma_1))$ and $\mathcal{U}(C^*(\Gamma_2))$ are topologically isomorphic as was proved in Example 5.3. □

As a consequence, we see that none of the invariants $\mathcal{U}(C^*(\Gamma))$ and $K_1(C^*(\Gamma))$, of a group algebra $C^*(\Gamma)$ is stronger than the other. The groups in Theorem 3.4 also show that two non-isometric (Abelian) C^*-algebras can have topologically isomorphic unitary groups *and* isomorphic K_1-groups. Take $\Phi_i = \Delta_i \times \mathbb{Z}_2$ with Δ_i defined as in Theorem 3.4. The same argument of Example 5.1 shows that $\mathcal{U}(C^*(\Delta_i)) \cong C(\Phi_i,\mathbb{R})\times\mathbb{T}^2\times\Delta_i\times\Delta_i$ and, hence, that $\mathcal{U}(C^*(\Phi_1)) \cong \mathcal{U}(C^*(\Phi_2))$. To see that $K_1(C^*(\Phi_1)) \cong K_1(C^*(\Phi_2))$ simply note that, by Lemma 5.4, $K_1(C^*(\Phi_i)) \cong K_1(C^*(\Delta_i))$ and that $K_1(C^*(\Delta_1)) \cong K_1(C^*(\Delta_2))$ by Theorem 3.4.

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